Welcome to the
International Iran Summer School on
Quantum Information 2008

Introduction to Topological Quantum Computation

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CSCM Center of excellence in Complex Systems & Condensed Matter

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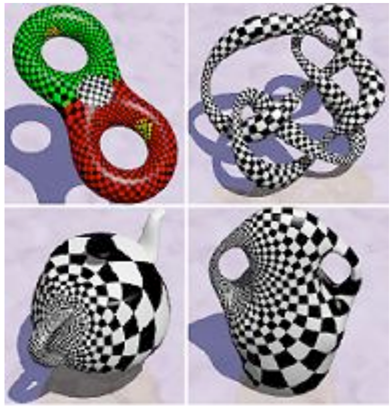

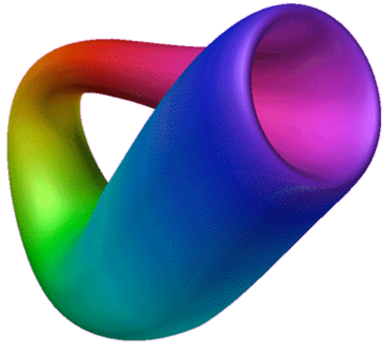

Also thanks to Laleh Memarzadeh, the Administrative Staff and Secretaries etc…

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"Optimal Resources for Topological Stabilizer Codes", Phys. Rev. A 76, 012305 (2007)

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H. Bombin and M. A. Martin-Delgado
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Homological Error Correction: Classical and Quantum Codes
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H. Bombin and M. A. Martin-Delgado
Phys. Rev. A **72**, 032313 (2005)

Outline of the Seminars

Lectures on Topological Effects in Quantum Information

→ **I. Introduction: Gaps And All That**

II. The Lieb-Mattis-Schulz Theorem

III. Non-Linear Sigma Model And Quantum Spin Chains

IV. Simple Models With topological Order In 1d: AKLT And Its Descendants

→ **V. Topological Orders And Quantum Information**

VI. Quantum Error Correction In The Stabilizer Formalism

→ **VII. Topological Stabilizer Codes**

→ **VIII. Topological Quantum Computation**

I. Introduction

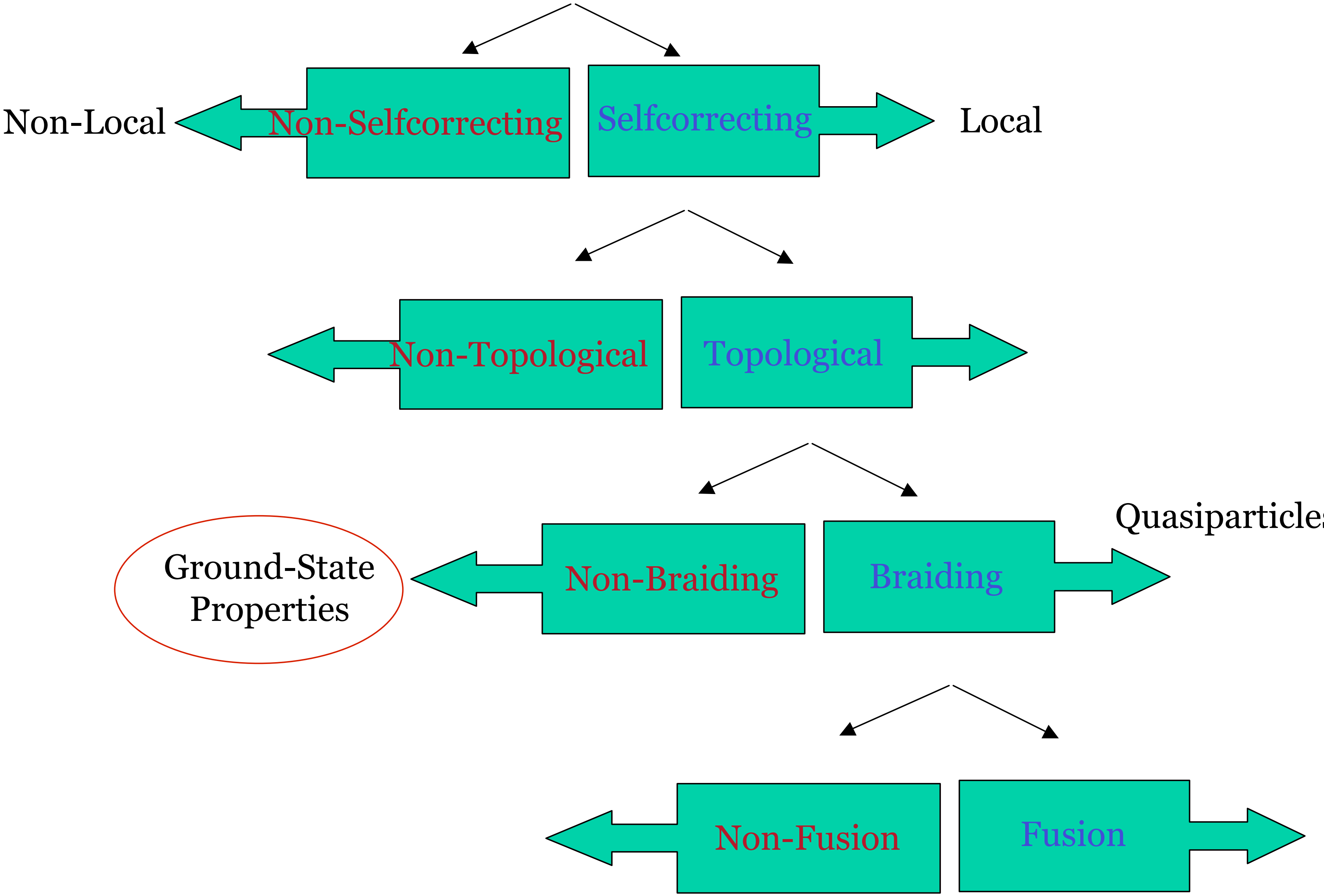
There are Many Paths towards the Topological Way to Quantum Computation

Topological Way = Alternative way to battle quantum decoherence

Let us framework our approach to topological Quantum information

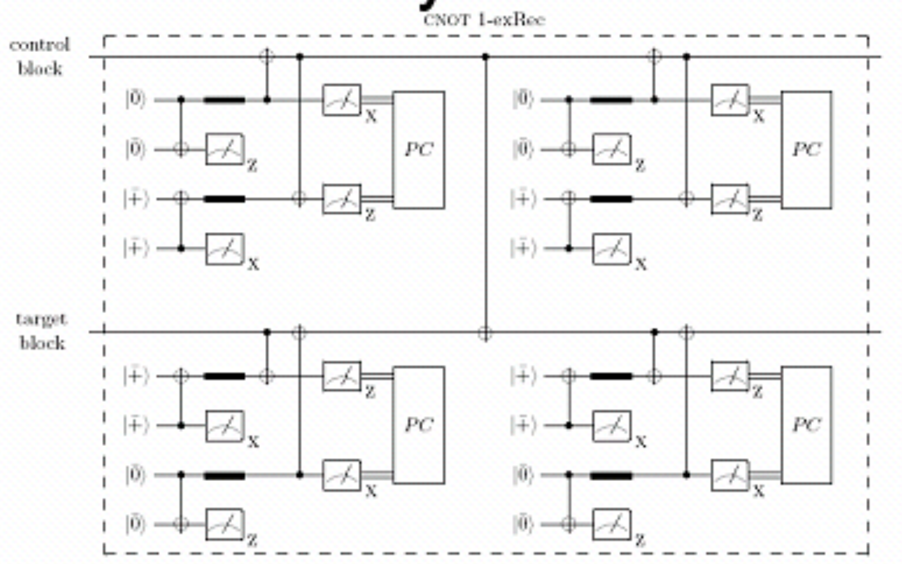
I. Introduction

The Fault-Tolerant Taxonomy



III.Quantum Error Correction

Lower bound on the accuracy threshold



A ***good*** gadget (one with sparse faults) is ***correct*** (simulates the ideal gate accurately).

For each of the level-1 extended Rectangles in a universal set, e.g. for the [[7,1,3]] (Steane) code, we can count the number of pairs of malignant locations; the CNOT 1-exRec dominates the threshold estimate. We find a rigorous lower bound on the accuracy threshold for *adversarial independent stochastic noise:*

$$\varepsilon_0 > 2.73 \times 10^{-5}$$

(assuming parallelism, fresh ancillas, nonlocal gates, fast measurements, fast and accurate classical processing, no leakage).

III.Quantum Error Correction

- **Bad News: the threshold is very small**

- **Good News: Fault-Tolerant Qomputation is possible**

Caution: the proof is constructive, there could be better thresholds

III.Quantum Error Correction

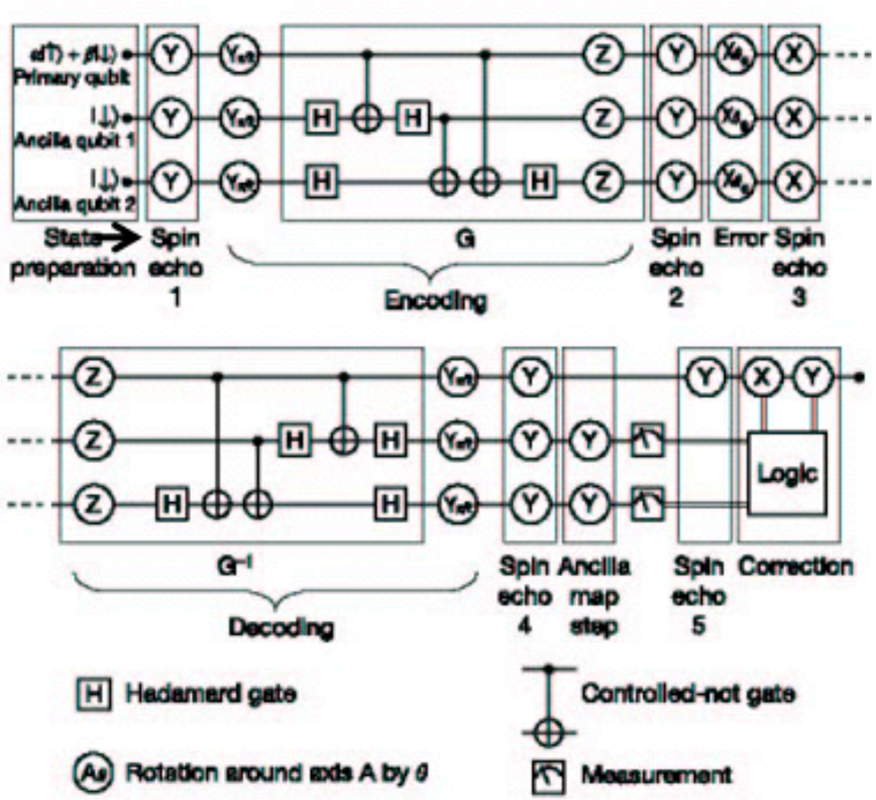
A realization of quantum error correction

J. Chiaverini *et al.*, [*Nature* **432**, 602-605 (2004)] implemented a three-qubit quantum repetition code using trapped ions. They prepared the encoded $|\bar{\psi}\rangle = a|\bar{0}\rangle + b|\bar{1}\rangle$ state, simulated noise that flips each qubit with probability ε, measured the error syndrome, and corrected the error.

The probability P of an encoded error was found to be

$$P = c + 2.6|ab|^2\,\varepsilon^2$$

… i.e., quadratic in ε.



Realization of quantum error correction

J. Chiaverini[1], D. Leibfried[1], T. Schaetz[1*], M. D. Barrett[1*], R. B. Blakestad[1], J. Britton[1], W. M. Itano[1], J. D. Jost[1], E. Knill[2], C. Langer R. Ozeri[1] & D. J. Wineland[1]

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* Present addresses: Max Planck Institut für Quantenoptik, Garching, Germany (T.S.); Physics Department, University of Otago, Dunedin, New Zealand (M.D.B.)

III. The Summer School

Topological Quantum Computation

The Topological Way to
Battle Decoherence

**Be Imaginative: Look for Alternatives
Against Decoherence**

(Remark: Decoherence is not always bad,
We are here because of decoherence)

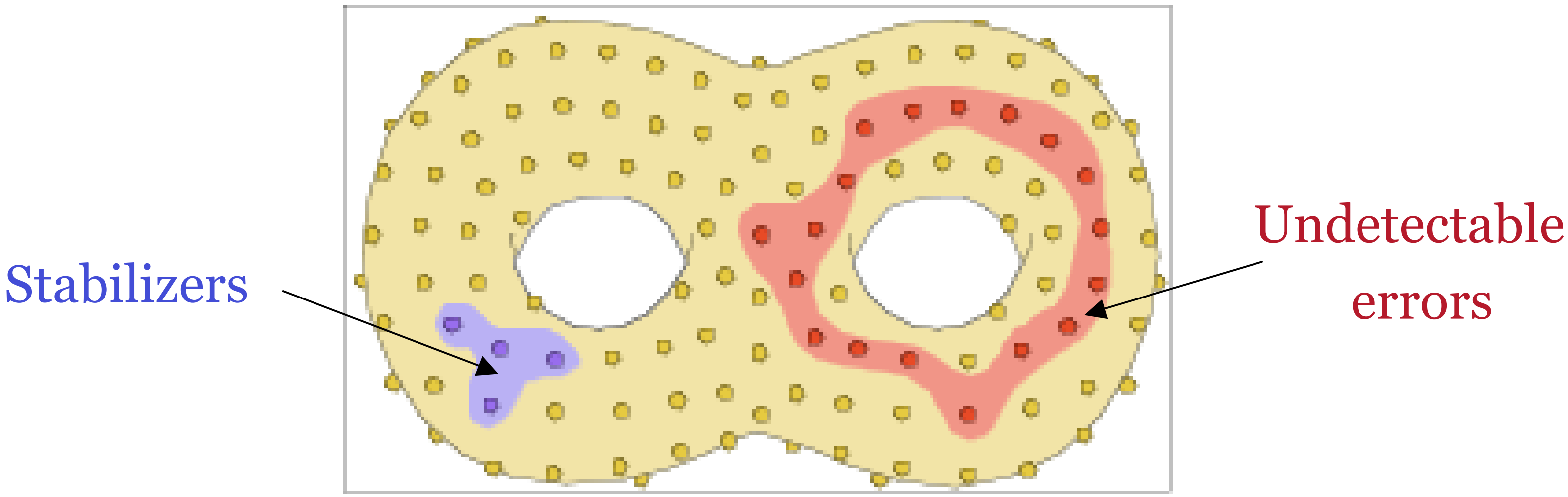
IV. 2-Colexes

Some relevant properties of these Quantum Lattice Hamiltonians

- They are local: interactions between nearest-neighbour qubits
- The Ground State is Degenerate and it is the Stabilizer Code
- The Ground State Degeneracy depends on the Topology of the Surface
- There is a Gap in the Spectrum separating the Ground State from the rest of Excited States

III.Topological Stabilizer Codes

- In order to introduce the idea of a topological stabilizer code (TSC), we must consider a topological space in which our physical qubits are to be placed, for example a surface.
- A TSC is a stabilizer code in which the generators of the stabilizer are **local** and undetectable errors (or encoded operators) are **topologically nontrivial**.



II.Stabilizer Codes

- A **stabilizer code**[1] C of length n is a subspace of the Hilbert space of a set of n qubits. It is defined by a stabilizer group S of Pauli operators, i.e., tensor products of Pauli matrices.

- It is enough to give the **generators** of S. For example:

$$|\psi\rangle \in \mathcal{C} \iff \forall\, s \in \mathcal{S} \quad s|\psi\rangle = |\psi\rangle$$

- Operators O that belong to the **normalizer** of S

$$\{ZXXZI, IZXXZ, ZIZXX, XZIZX\}$$

leave invariant the code space C. If they do not belong to the stabilizer, then they act non-trivially in the code subspace.

$$O \in \mathbf{N}(\mathcal{S}) \iff O\mathcal{S} = \mathcal{S}O$$

[1] D. Gottesman 95

II.Stabilizer Codes

- A encoded state can be subject to **errors**.
- To correct them, we measure a set of generators of *S*. The results of the measurement compose the **syndrome** of the error. Errors can be corrected as long as the syndrome lets us distinguish among the possible errors.
- Since correctable errors always form a vector space, it is enough to consider Pauli operators, which form a basis.
- We say that a Pauli error *e* is **undetectable** if it belongs to **N**(*S*)-*S*. In such a case, the syndrome says nothing:

$$\forall s \in \mathcal{S} \qquad s\, e|\psi\rangle = e\, s\mathbf{0}|\psi\rangle = e|\psi\rangle$$

- A set of Pauli errors *E* is correctable iff:

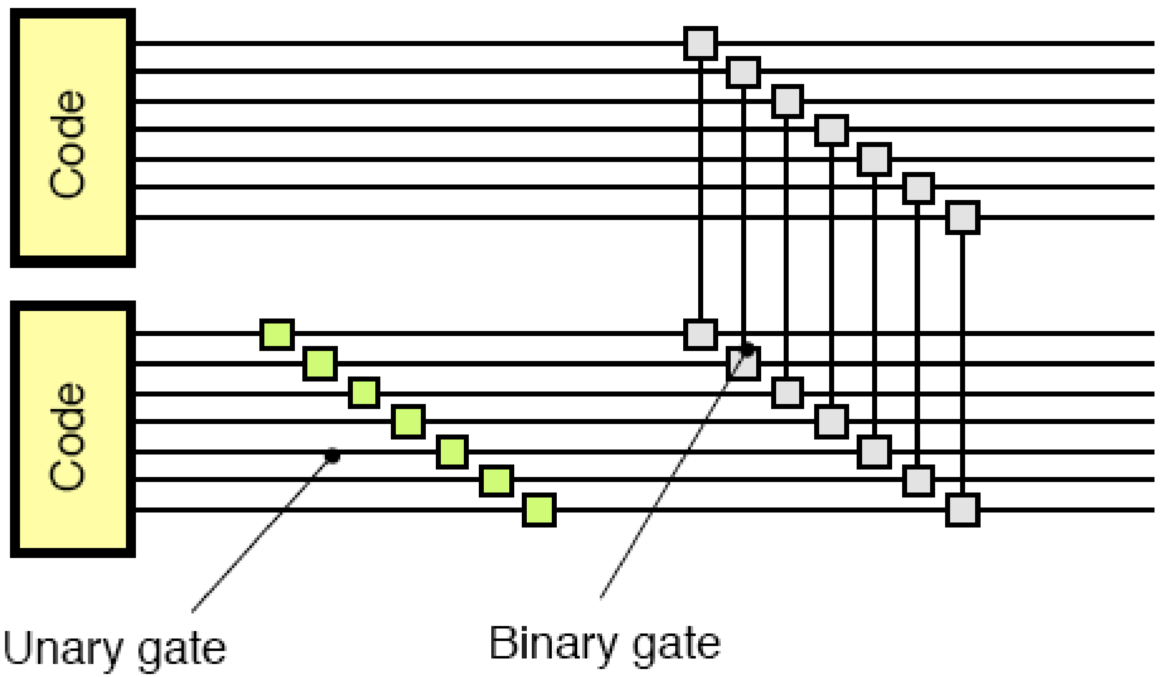
$$E^{\dagger}E \cap \mathcal{N}(\mathcal{S}) \in \mathcal{S}.$$

II.Stabilizer Codes

- A **stabilizer code**[1] C of length n is a subspace of the Hilbert space of a set of n qubits. It is defined by a stabilizer group S of Pauli operators, i.e., tensor products of Pauli matrices.

$$|\psi\rangle \in \mathcal{C} \qquad \Longleftrightarrow \qquad \forall\, s \in \mathcal{S} \quad s|\psi\rangle = |\psi\rangle$$

- Some stabilizer codes are specialy suitable for quantum computation. They allow to perform operations in a **transversal** and **uniform** way:



[1] D. Gottesman 95

II.Stabilizer Codes

Gate Sets

- Several codes allow the transversal implementation of

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad K = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad \Lambda = \begin{pmatrix} I_2 & 0 \\ 0 & X \end{pmatrix}$$

  which generate the **Clifford group**. This is useful for quantum information tasks such as teleportation or **entanglement distillation**.
- Quantum **Reed-Muller** codes[1] are very special. They allow **universal computation** through transversal gates

$$K^{1/2} = \begin{pmatrix} 1 & 0 \\ 0 & i^{1/2} \end{pmatrix} \qquad \Lambda = \begin{pmatrix} I_2 & 0 \\ 0 & X \end{pmatrix}$$

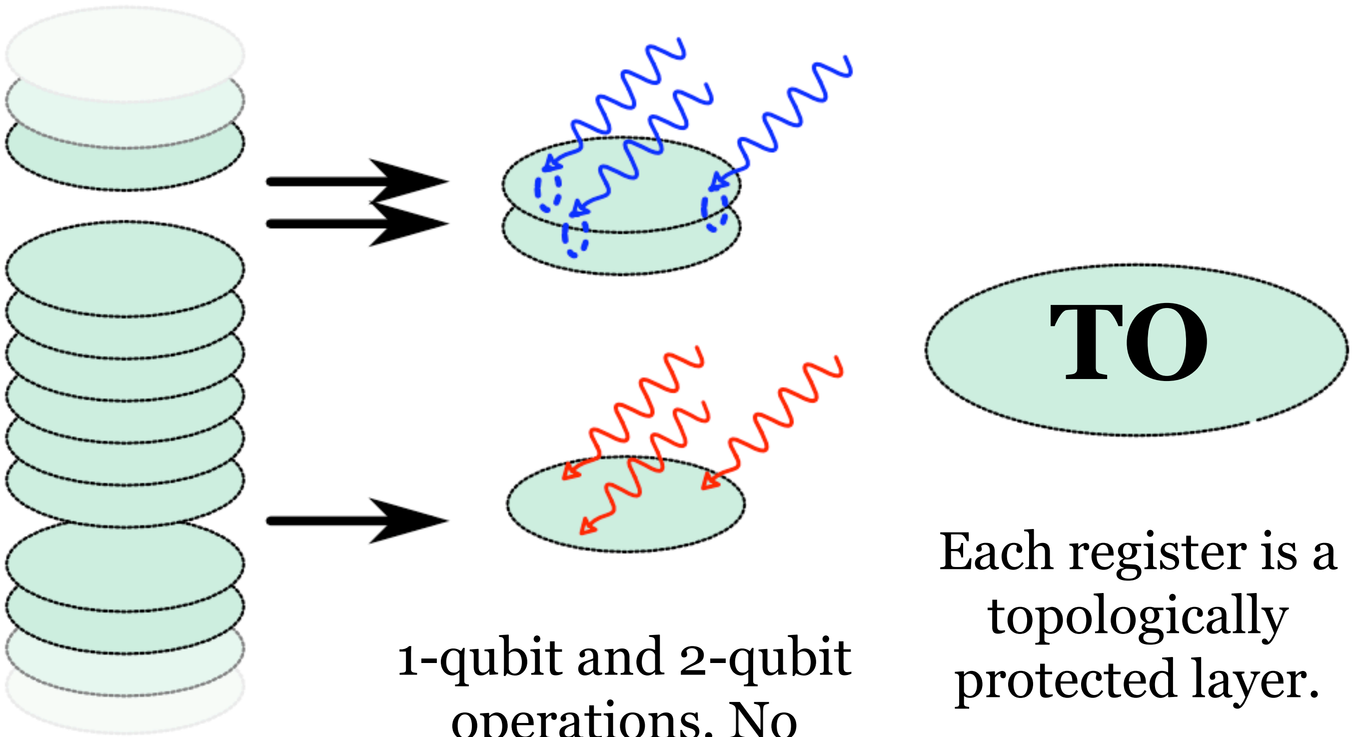
  and transversal measurements of X and Z.
- We will see how both sets of operations can be transversally implemented in 2D and 3D topological color codes:

Color Codes = Transversality + Topology

[1] E. Knill *et al.*

IV. 2-Colexes

- **Goal**: 2-dimensional layers as quantum registers, protected by TO. Operations on encoded qubits without selective addressing of physical qubits.



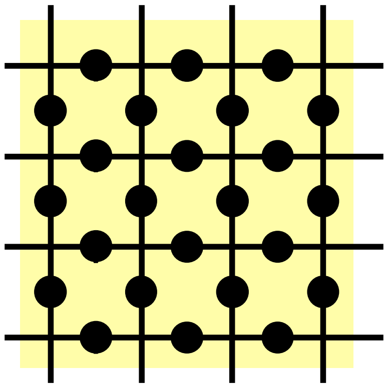
Register stack

1-qubit and 2-qubit operations. No addressing.

Each register is a topologically protected layer.

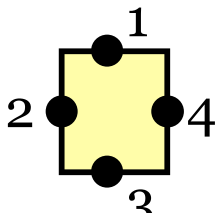
III.Topological Stabilizer Codes

- The first example of TSC were **surface codes**[1], which are based on Z_2 homology and cohomology.



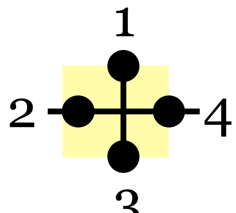
1 qubit per link

Face operator



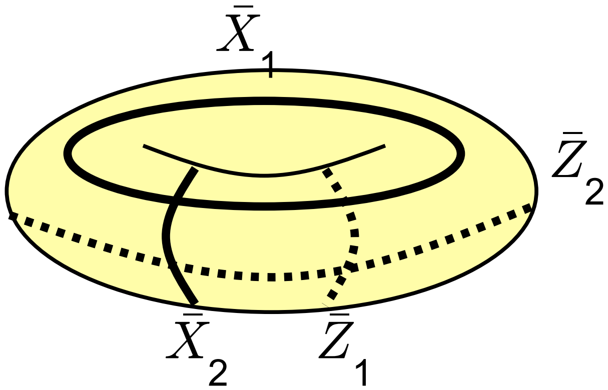
$X_1 X_2 X_3 X_4$

Vertex operator

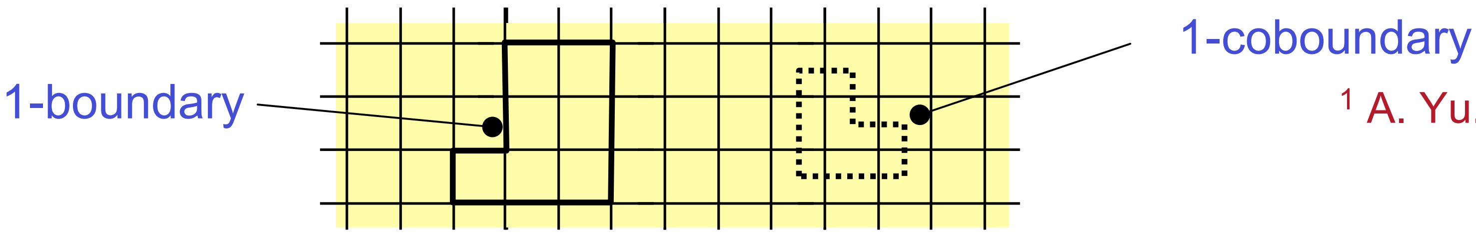


$Z_1 Z_2 Z_3 Z_4$

Encoded operator basis



- S gets identified with 1-boundaries and 1-coboundaries, and $\mathbf{N}(S)$ with 1-cycles and 1-cocycles.



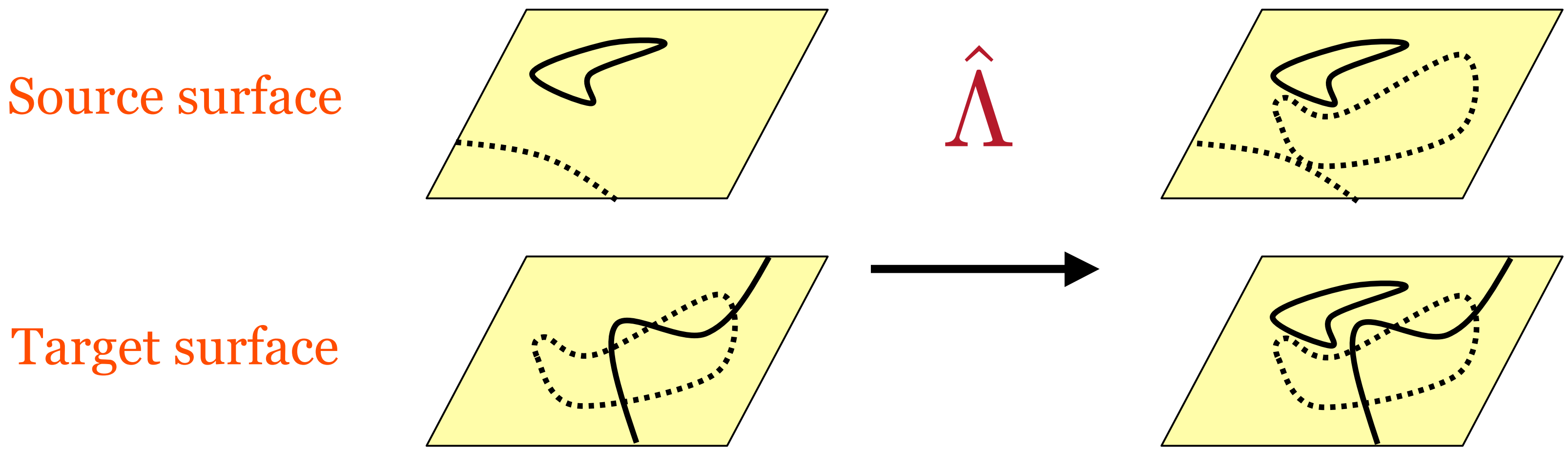
[1] A. Yu. Kitaev 97

III.Topological Stabilizer Codes

- The **CNot gate** can be implemented transversally on surface codes. First, its action under conjugation on operators is:

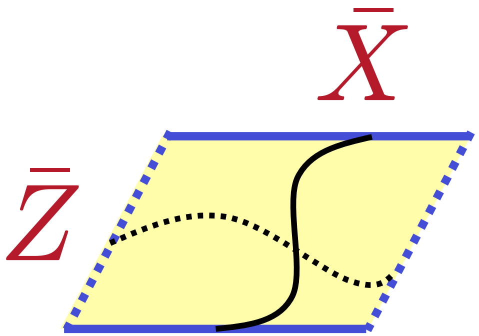
$$\Lambda : \quad \begin{array}{ll} IX \longrightarrow IX & IZ \longrightarrow ZZ \\ XI \longrightarrow XX & ZI \longrightarrow ZI \end{array}$$

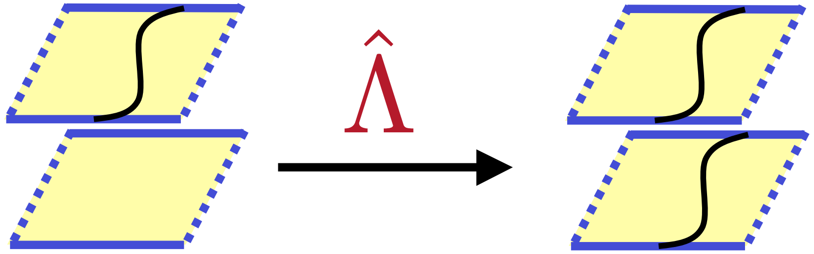
- Thus the transversal action of the CNot on a surface code, at the level of operators, is simply to copy chains forward and cochains backwards.



III.Topological Stabilizer Codes

- Finally, to see the action of the **tranversal CNOT on the code**, we have to choose a Pauli basis for the encoded qubits. In the simplest example we have a single qubit in a square surface with suitable borders:



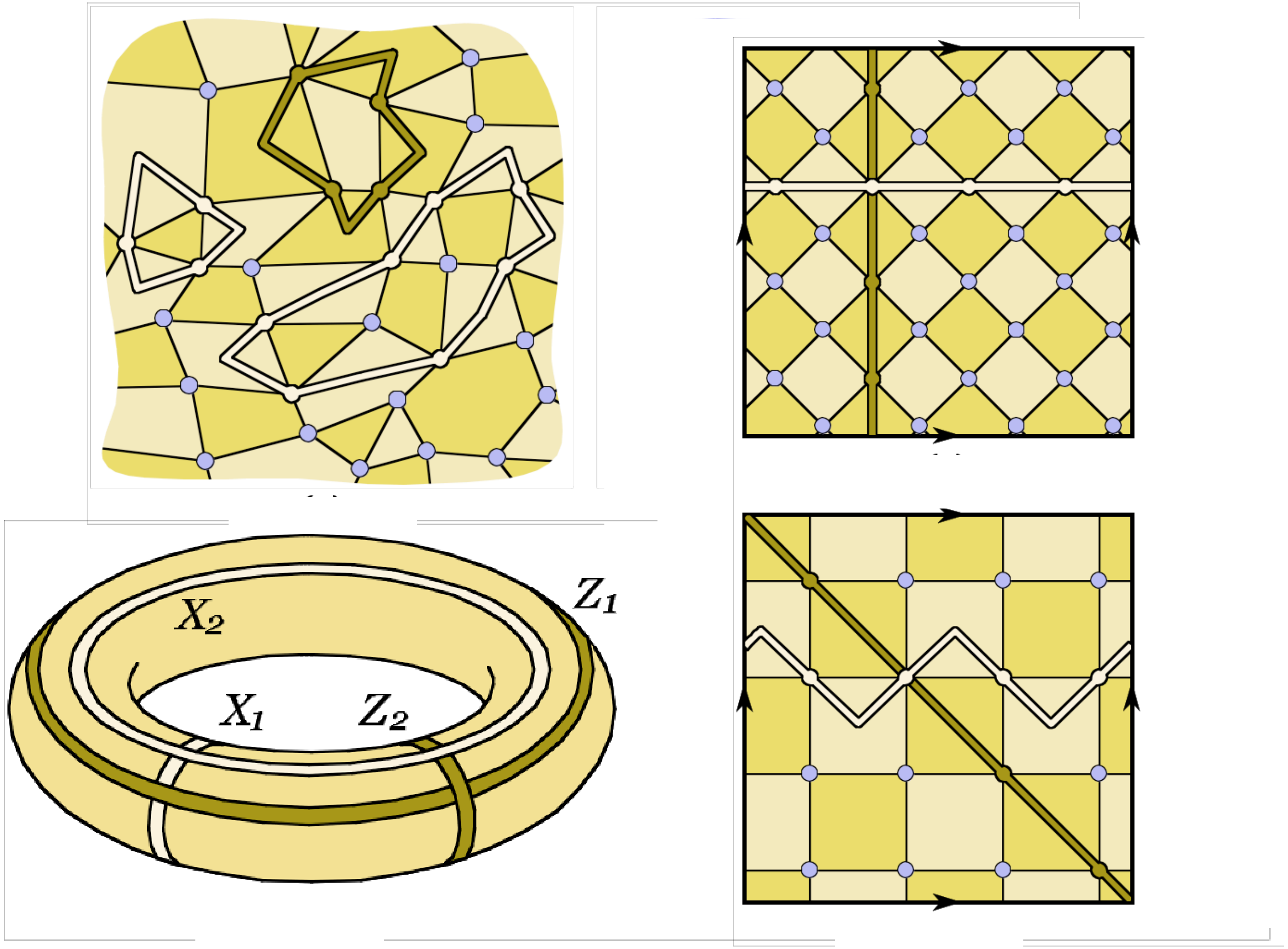


$\hat{\Lambda}\,\bar{X}\,\hat{I}\,\hat{\Lambda}^{+} = \bar{X}\,\bar{X}$

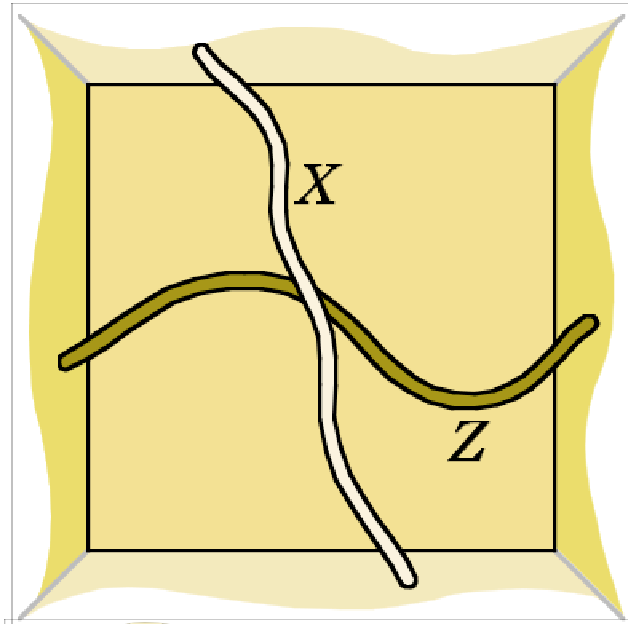
- Clearly **the action of a transversal CNot is itself a CNot gate on the encoded qubits**. However, this is the only gate we can get with surface codes. If we want to get further, **we have to go beyond homology**.

III.Topological Stabilizer Codes

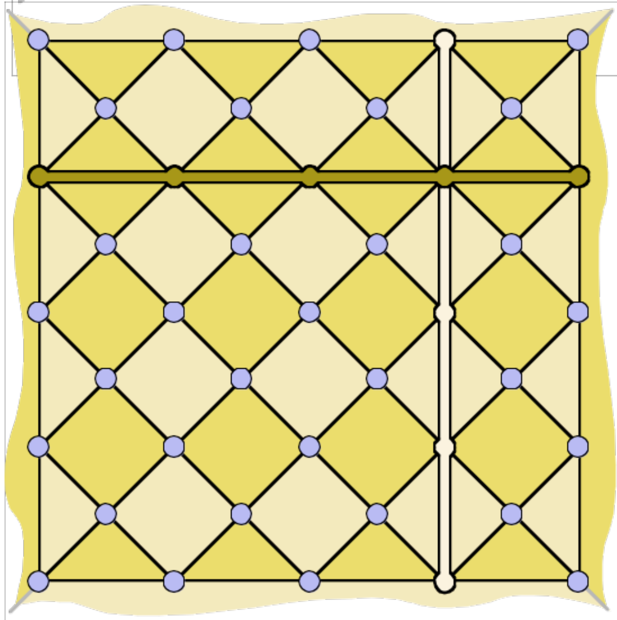
A surface code (Kitaev) from another perspective:

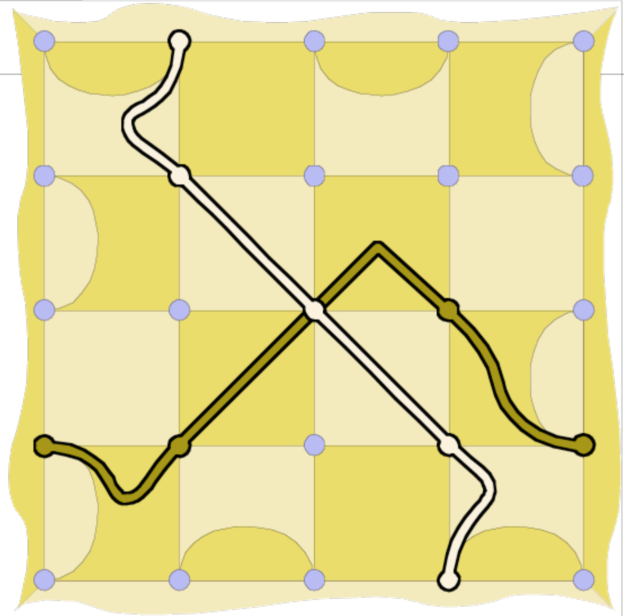


III.Topological Stabilizer Codes



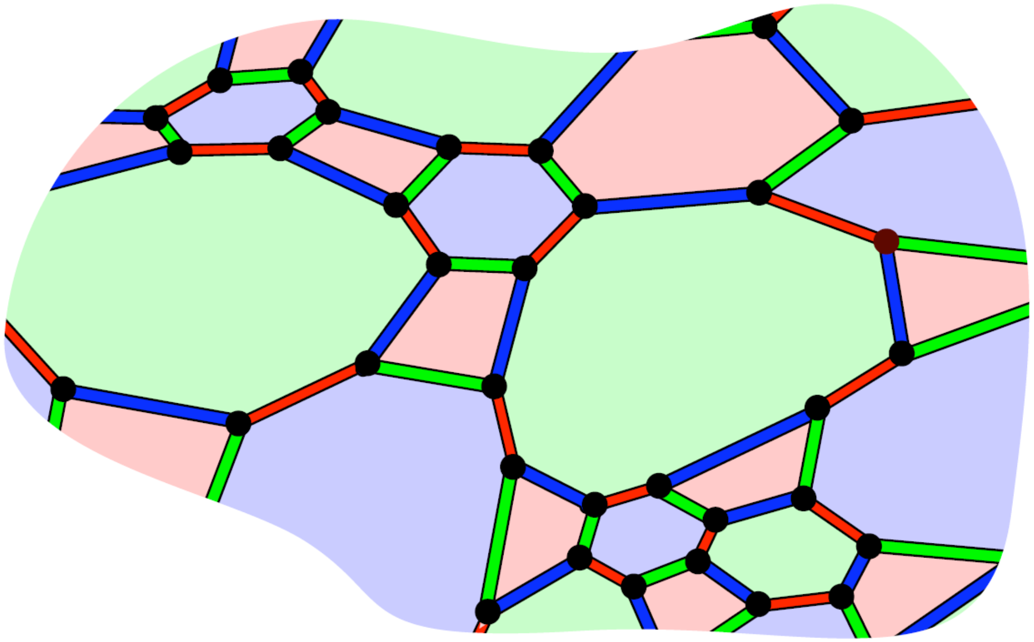
Single Qubit





IV. 2-Colexes

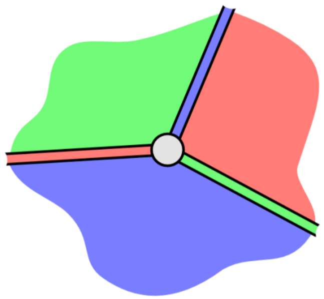
- A 2-colex is a **trivalent** 2-D lattice with **3-colored** **faces**.

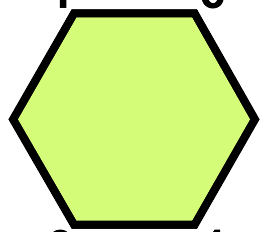


- **Edges can be 3-colored accordingly**. Blue edges connect blue faces, and so on.
- The name 'colex' is for 'color complex'. *D*-colexes of arbitrary dimension can be defined. Their key feature is that the whole structure of the complex is contained in the 1-skeleton and the coloring of the edges.

IV. 2-Colexes

- To construct a **color code** from a 2-colex, we place 1 qubit at each **vertex** of the lattice. The generators of S are **face operators**:





$$B_f^X = X_1 X_2 X_3 X_4 X_5 X_6$$

$$B_f^Z = Z_1 Z_2 Z_3 Z_4 Z_5 Z_6$$

- Transversal Clifford gates should belong to **N**(S). We have:

$$\hat{H} B_f^X \hat{H}^\dagger = B_f^Z \qquad \hat{K} B_f^X \hat{K}^\dagger = (-)^{\frac{v}{2}} B_f^X B_f^Z$$

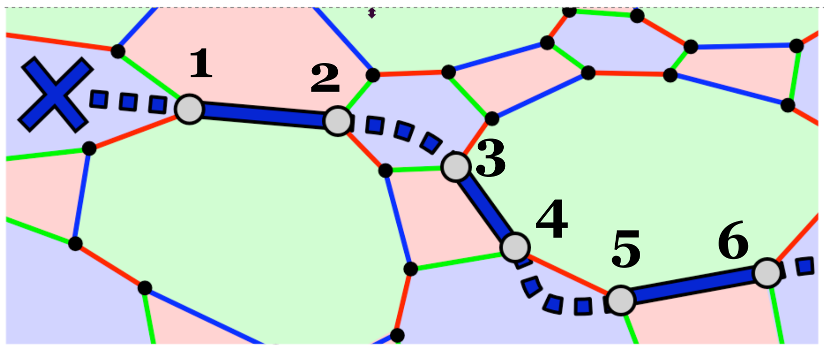
$$\hat{H} B_f^Z \hat{H}^\dagger = B_f^X \qquad \hat{K} B_f^Z \hat{K}^\dagger = B_f^Z$$

- Here v is the number of vertices in the face. If it is a multiple of 4 for every face, then K is in **N**(S). H always is.
- As for the CNot gate, it is clearly in **N**(S) (it is a CSS code).

IV. 2-Colexes

- In order to understand 2-D color codes, we have to introduce **string operators** in the picture. As in surface codes, we play with Z_2 homology. However, there is a **new ingredient**, **color**.
- A **blue string** is a collection of **blue links**

Endpoint



String operators

$$S^X = X_1 X_2 X_3 X_4 X_5 X_6 \dots$$

$$S^Z = Z_1 Z_2 Z_3 Z_4 Z_5 Z_6 \dots$$

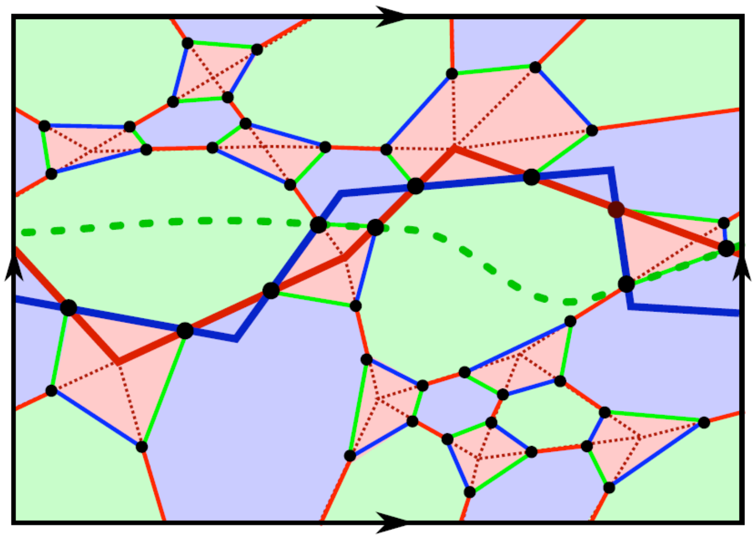
(hexagonal bishop with flavor X or Z):

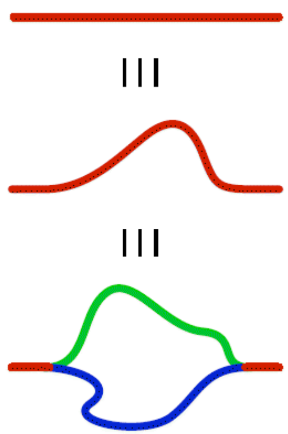
- Strings can have endpoints, located at faces of the same color. However, in that case the corresponding string and face operators will not commute. Therefore, a string operator belongs to **N**(S) iff the string has no endpoints.

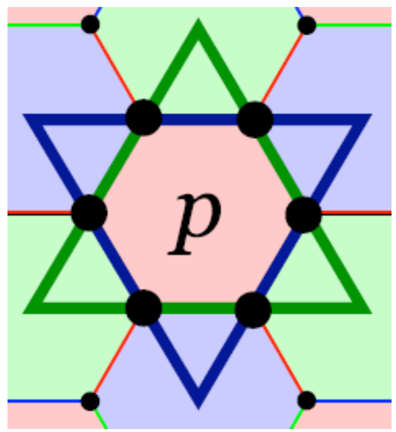
IV. 2-Colexes

Continous Visualization of Color Strings

- For each color we can form a **shrunk graph**. The red one is:







Red faces ⟶ vertices

Red edges ⟶ edges

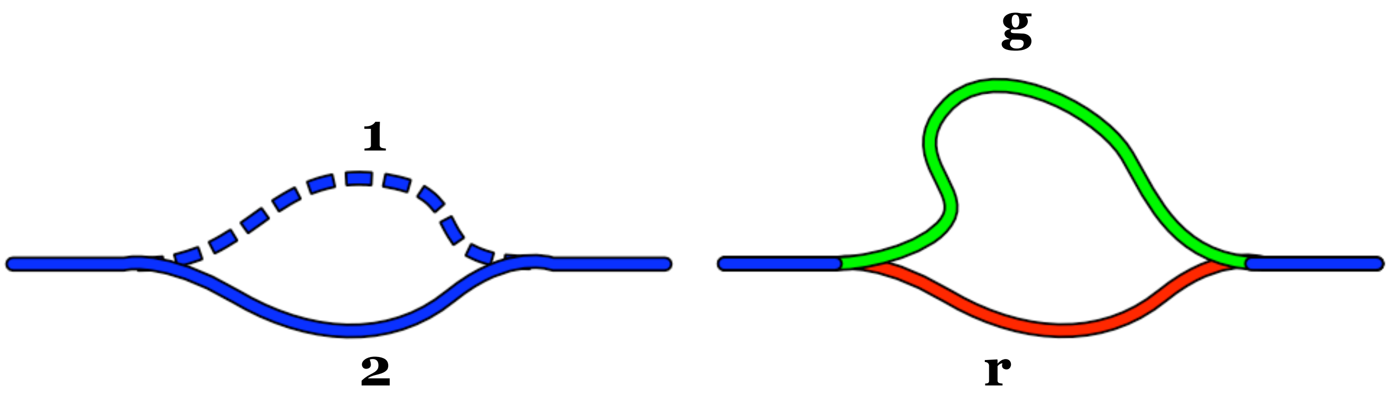
Blue and green faces ⟶ faces, faces

A red face is also blue or green string

- Thus for each color homology works as in surface codes. The new feature is the possibility to **combine** homologous blue and red string operators of the same kind to get a green one.

IV. 2-Colexes

- Strings can be **deformed** and colors **branched**:



$$S_1^\sigma \sim S_2^\sigma \qquad S_r^\sigma S_g^\sigma \sim S_b^\sigma$$

Equivalent strings act equally on the Ground State.

IV. 2-Colexes

- Since there are two independent colors, the number of encoded qubits should double that of a surface code. Lets check this for a surface **without boundary** using the Euler characteristic for any *shrunk* lattice. $\chi = V + F - E$
- Face operators are subject to the **conditions**

$$\prod_{f\in {\color{red}\bullet}} B_f^\sigma = \prod_{f\in {\color{blue}\bullet}} B_f^\sigma = \prod_{f\in {\color{green}\bullet}} B_f^\sigma ,$$

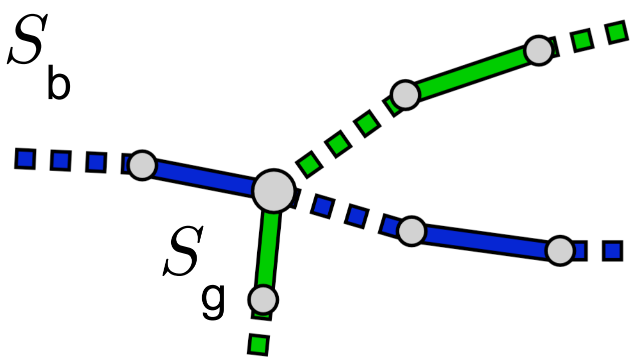
so that the total number of generators is $g = 2(F + V - 2)$

- The number of physical qubits is $n = 2E$. Therefore the number of encoded qubits q is twice the first Betti number of the manifold:

$[[n, k, d]]$ $\qquad$ $k = n - g = 4 - 2\chi = 2h_1$

IV. 2-Colexes

- In order to form a **Pauli basis** for the operators acting on encoded qubits, we can use as in surface codes those string operators (SO) that are not homologous to zero.
- To this end, we need the commutation rules for SO.
- Clearly SO of the same type (X or Z) always commute.
- A string is made up of edges with two vertices each. Therefore, two SO of the same color have an even number of qubits in common an they commute.
- SO of **different colors** can **anticommute**, but only if they **cross** an odd number of times:

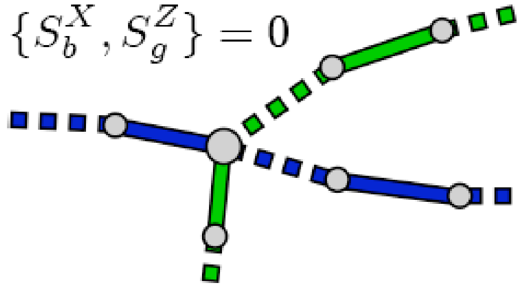


$$\{S_b^X, S_g^Z\} = 0$$

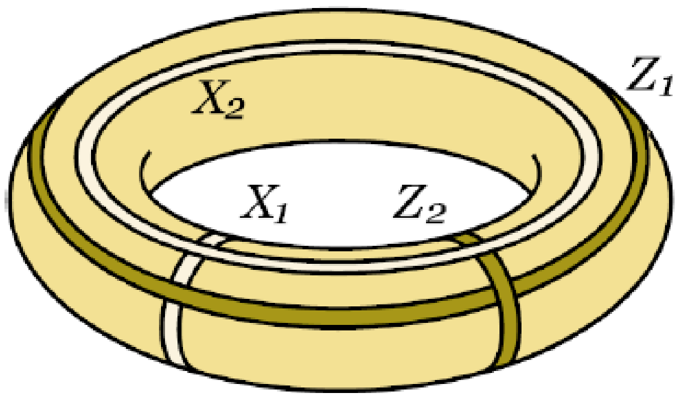
IV. 2-Colexes

String Operators

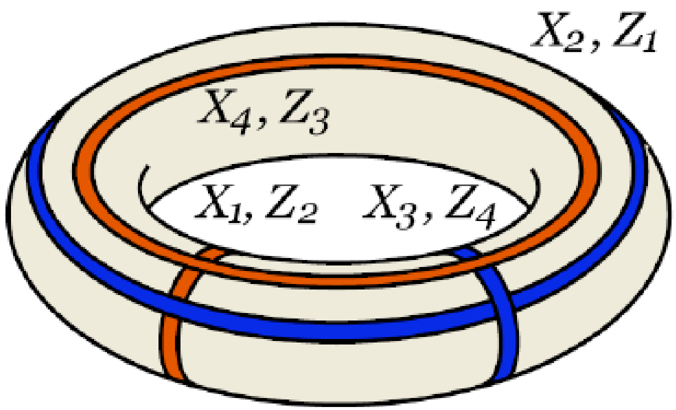
- For each colored string S, there are a **pair** of **string operators**, S^X and S^Z, products of Xs or Zs along S.
- String operators either commute or anticommute.
- Two string operators **anticommute** when they have **different color and type** and **cross** an odd number of times.



- As in surface codes, encoded X and Z operators can be chosen from closed string operators which are not boundaries.
- The number of **encoded qubits** is **twice** as in a surface code:



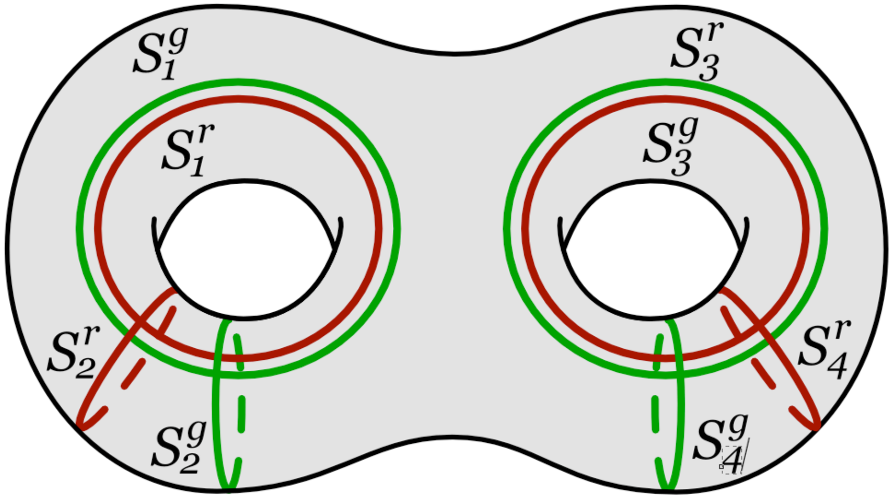
Surface code: 2 qubits



Color code: 4 qubits

IV. 2-Colexes

- Now we can construct the desired operator basis for the encoded qubits. In a 2-torus a possible choice is:



$$
\begin{array}{ll}
S_1^{gX} \leftrightarrow X_1 & S_2^{rZ} \leftrightarrow Z_1 \\
S_2^{rX} \leftrightarrow X_2 & S_1^{gZ} \leftrightarrow Z_2 \\
S_2^{gX} \leftrightarrow X_3 & S_1^{rZ} \leftrightarrow Z_3 \\
\vdots & \vdots
\end{array}
$$

$$X_i Z_j = (-1)^{\delta_{i,j}} Z_j X_i$$

Encoded qubits = $2h_1$

h_1 = first Betti number

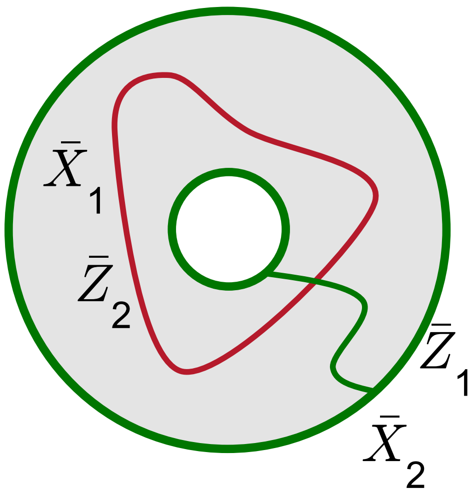
- **However**, if we apply the transversal H gate to such a code the resulting encoded gate is not H. The underlying reason is that for a string S we **never** have

$$\{S_b^X, S_g^Z\} = 0$$

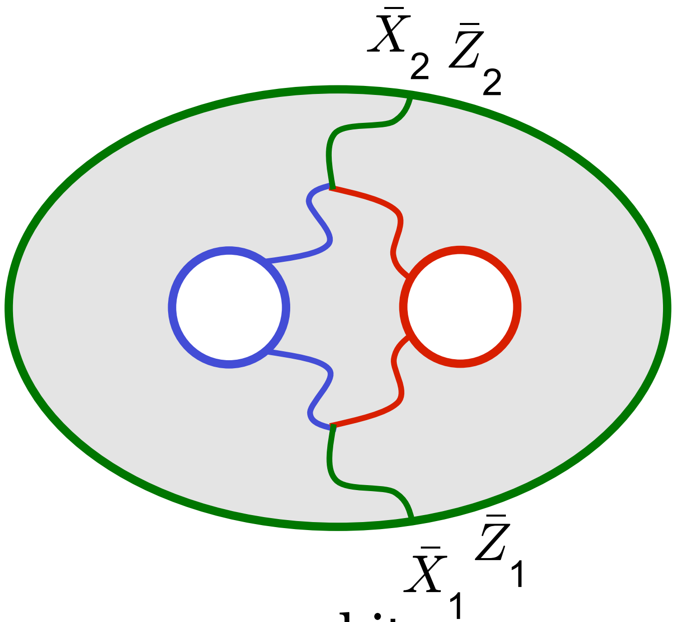
IV. 2-Colexes

Way out:

- But we can consider surfaces with **boundary**. To this end, we take a sphere, which encodes no qubit, and **remove faces**.
- When a face is removed, the resulting boundary must have its color, and only strings of that color can end at the boundary.

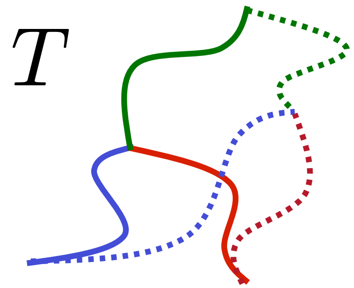


2 qubits



2 qubits

Toy Baryon or String-Net[1]



$$\{T^X, T^Z\} = 0$$

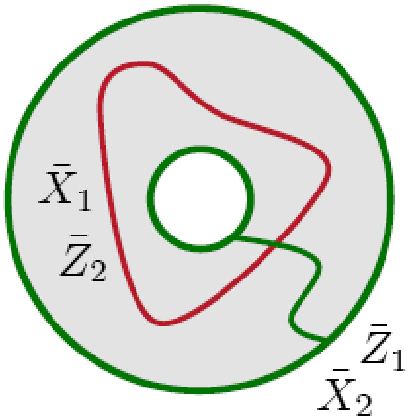
As desired!

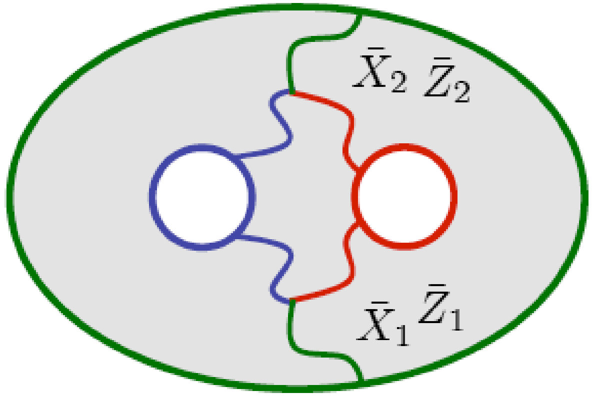
[1]Wen et al.

IV. 2-Colexes

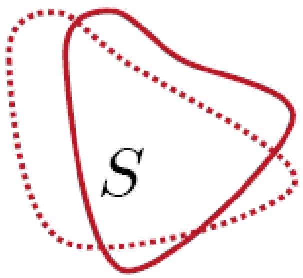
Borders and String-Nets

- Borders are big missing plaquettes. Their color is that of the erased plaquette.



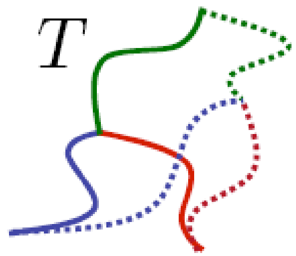


- Both examples encode 2 qubits, but the second requires **string-net operators**.
- These have a new feature, which turns out to be crucial in orther to be able to implement transversally the whole Clifford group:



$$[S^X, S^Z] = 0$$

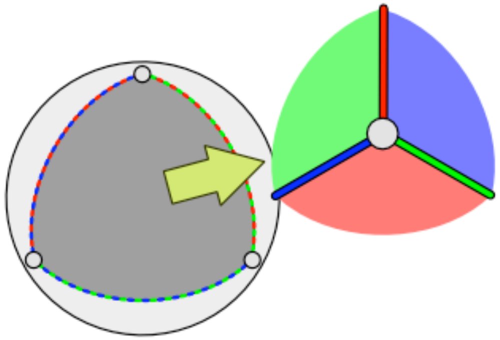
$$\{T^X, T^Z\} = 0$$

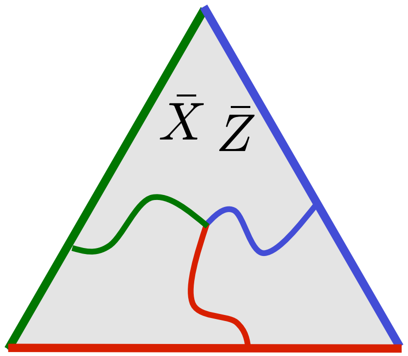


IV. 2-Colexes

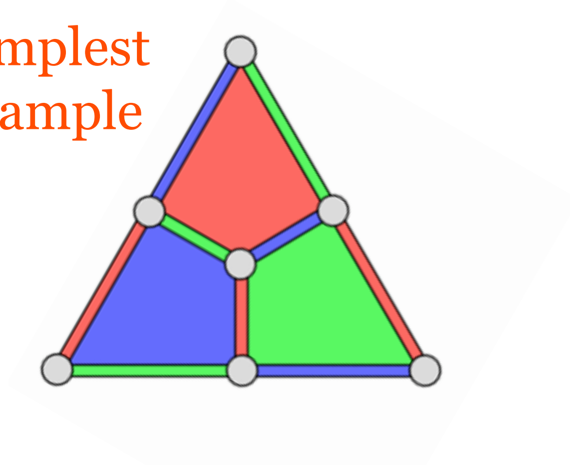
Look for 2-colexes with string-nets:

- We can even encode a single qubit an remove the need for holes. If we remove a site and neighboring links and faces from a 2-colex in a **sphere**, we get a **triangular** code:

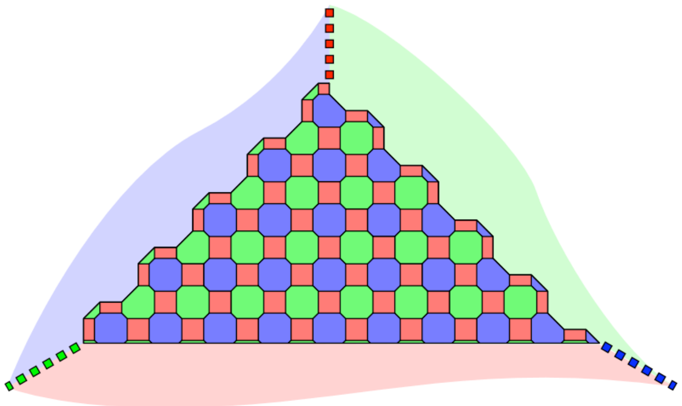




Simplest example



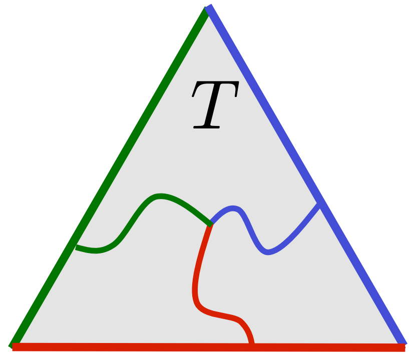
- We can construct triangular codes of arbitrary sizes. The vertices per face can be 4 and 8 so that K is in $\mathbf{N}(S)$.



IV. 2-Colexes

- The transversal H clearly amounts to an encoded H:

$$H: \begin{array}{l} X \longrightarrow Z \\ Z \longrightarrow X \end{array} \qquad \hat{H}: \begin{array}{l} T_X \longrightarrow T_Z \\ T_Z \longrightarrow T_X \end{array}$$



- This is also true for K. The anticommutation properties of T imply that its support consists of an odd number of qubits:

$$K: \begin{array}{l} X \longrightarrow iXZ \\ Z \longrightarrow Z \end{array} \qquad \hat{K}: \begin{array}{l} T_X \longrightarrow \pm i T_X T_Z \\ T_Z \longrightarrow T_Z \end{array}$$

- Therefore, the **Clifford group** can be implemented transversally in triangular codes.

IV. 2-Colexes

Triangular Codes

- Encoded ***X* and *Z*** operators:

$$\hat{X} = X^{\otimes n} \quad \hat{Z} = Z^{\otimes n} \qquad \{\hat{Z}, \hat{X}\} = 0$$

n = # physical qubits.

$$[\hat{X}, B_f^Z] = 0, \quad [\hat{Z}, B_f^X] = 0$$

- The **Clifford group** is implemented with global operators:

$$\hat{H} = H^{\otimes n} \qquad \hat{K} = K^{\otimes n} \qquad \hat{\Lambda} = \Lambda^{\otimes n}$$

$$\hat{H}\hat{X}\hat{H}^\dagger = \hat{Z} \qquad \hat{K}\hat{X}\hat{K}^\dagger = \pm i\hat{X}\hat{Z} \qquad \hat{\Lambda}\widehat{IX}\hat{\Lambda}^\dagger = \widehat{IX}, \quad \hat{\Lambda}\widehat{XI}\hat{\Lambda}^\dagger = \widehat{XX}$$

$$\hat{H}\hat{Z}\hat{H}^\dagger = \hat{X} \qquad \hat{K}\hat{Z}\hat{K}^\dagger = \hat{Z} \qquad \hat{\Lambda}\widehat{IZ}\hat{\Lambda}^\dagger = \widehat{ZZ}, \quad \hat{\Lambda}\widehat{ZI}\hat{\Lambda}^\dagger = \widehat{ZI}$$

$$\hat{H}B_f^X\hat{H}^\dagger = B_f^Z \qquad \hat{K}B_f^X\hat{K}^\dagger = B_f^X B_f^Z \qquad \hat{\Lambda}IB_f^X\hat{\Lambda}^\dagger = IB_f^X, \quad \hat{\Lambda}B_f^X I\hat{\Lambda}^\dagger = B_f^X B_f^X$$

$$\hat{H}B_f^Z\hat{H}^\dagger = B_f^X \qquad \hat{K}B_f^Z\hat{K}^\dagger = B_f^Z \qquad \hat{\Lambda}IB_f^Z\hat{\Lambda}^\dagger = B_f^Z B_f^Z, \quad \hat{\Lambda}B_f^Z I\hat{\Lambda}^\dagger = B_f^Z I$$

IV. 2-Colexes

•Quantum Hamiltonians and Topological Orders

•Given a Topological Stabilizer Code → Strongly Correlated System with Topological Order

•The Hamiltonian is constructed from the Code Generators: Several Forms

•Original Form for Kitaev Code

$$H = -\sum_{\mathcal{O}\in\mathcal{S}} \mathcal{O} = -\sum_{p\in P} Z_p - \sum_{v\in V} X_v$$

IV. 2-Colexes

- Checkerboard Forms

- Kitaev's Code

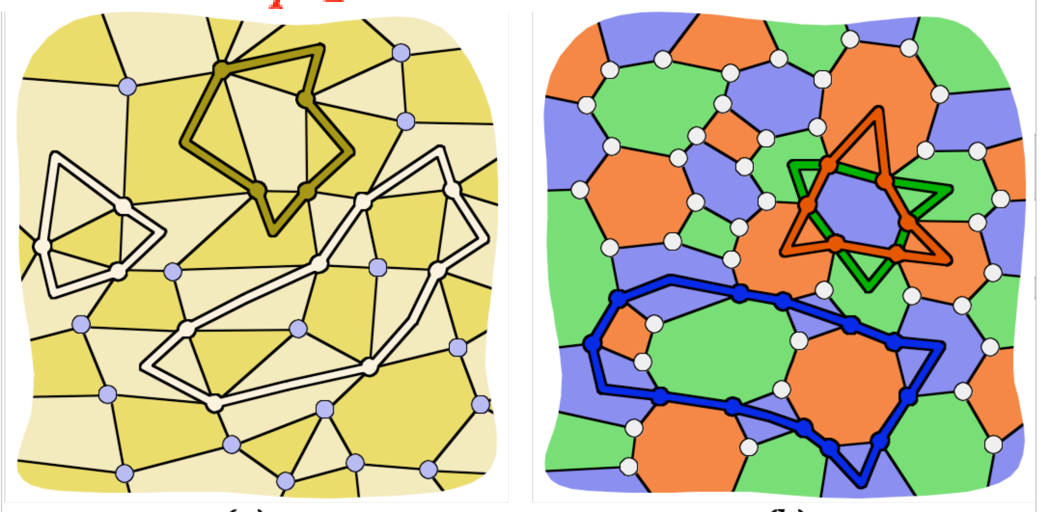
$$H_{\mathrm{K}} = -\sum_{p \in P_{\mathrm{D}}} B_p^X - \sum_{p \in P_{\mathrm{L}}} B_p^Z$$

Plaquettes: Separated

- Color Codes

$$H_{\mathrm{c}} = -\sum_{p \in P} (B_p^X + B_p^Z)$$

Plaquettes: Together



(a) (b)

IV. 2-Colexes

Some relevant properties of these Quantum Lattice Hamiltonians

- They are local: interactions between nearest-neighbour qubits

- The Ground State is Degenerate and it is the Stabilizer Code

- The Ground State Degeneracy depends on the Topology of the Surface

- There is a Gap in the Spectrum separating the Ground State from the rest of Excited States

IV. 2-Colexes

- **Ground State GS** can be described by applying string-net operators to the GS:

We can give an expression for the states of the logical qubit

$$\{|\bar{0}\rangle, |\bar{1}\rangle\}$$

:

$$|\bar{0}\rangle = \prod_{\mathrm{b}} (1 + B_b^X) \prod_{\mathrm{p}} (1 + B_p^X) |0\rangle^{\otimes n}$$

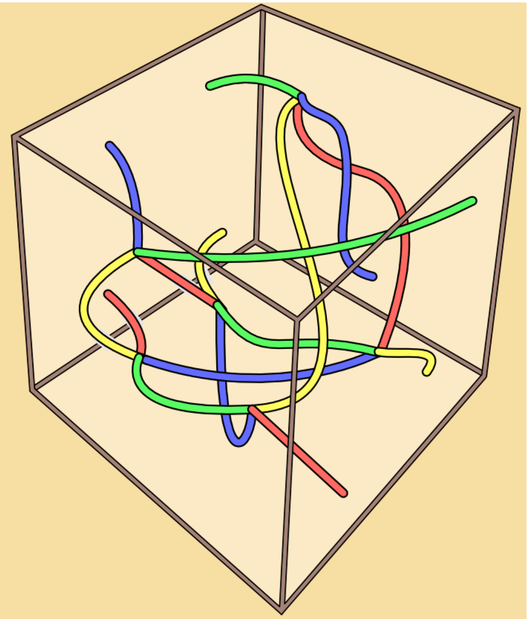
and

$$|\bar{1}\rangle := \hat{X}|\bar{0}\rangle$$

$$\hat{Z}|\bar{l}\rangle = (-1)^l |\bar{l}\rangle \qquad l = 0, 1$$

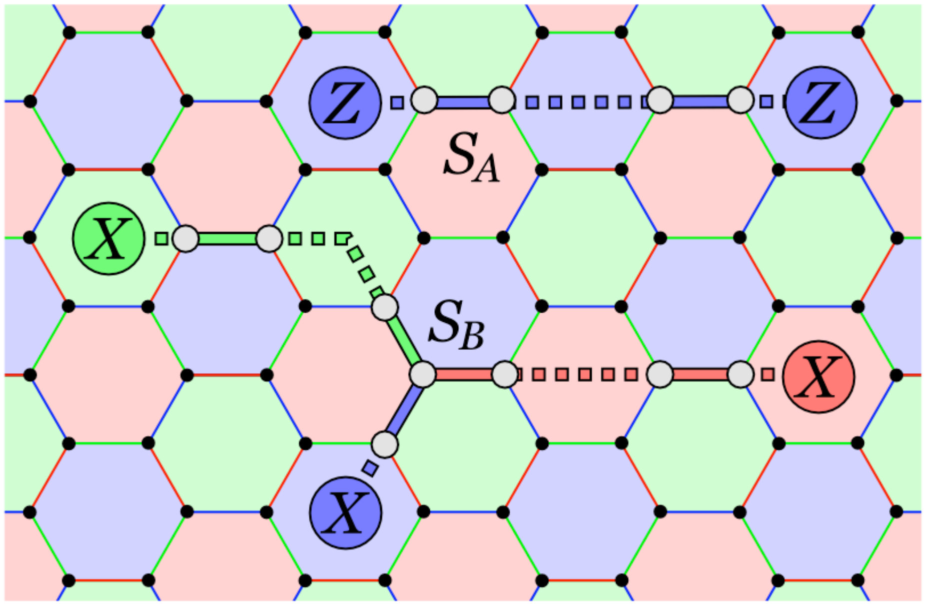
.

$$|\bar{0}\rangle = \sum_{\mathrm{string-nets}} B_s^X |0\rangle^{\otimes n}$$



IV. 2-Colexes

- **Excitations** can be created applying string operators to the GS:

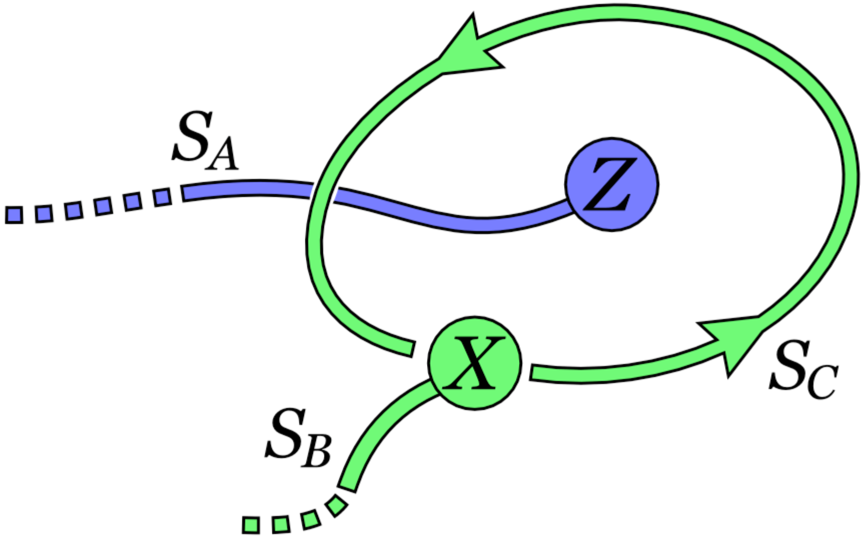


$$S^{z}_{A}\, S^{x}_{B}\, |\mathrm{GS}\rangle$$

Each endpoint is a quasiparticle, a violation of a face condition.

Anyons

- The quasiparticles that populate the system are **abelian anyons**.
- When, for example, a green X excitation loops around a blue Z excitation, the system gets a global **minus sign**:



$$\{S^{z}_{A}, S^{x}_{C}\} = 0$$

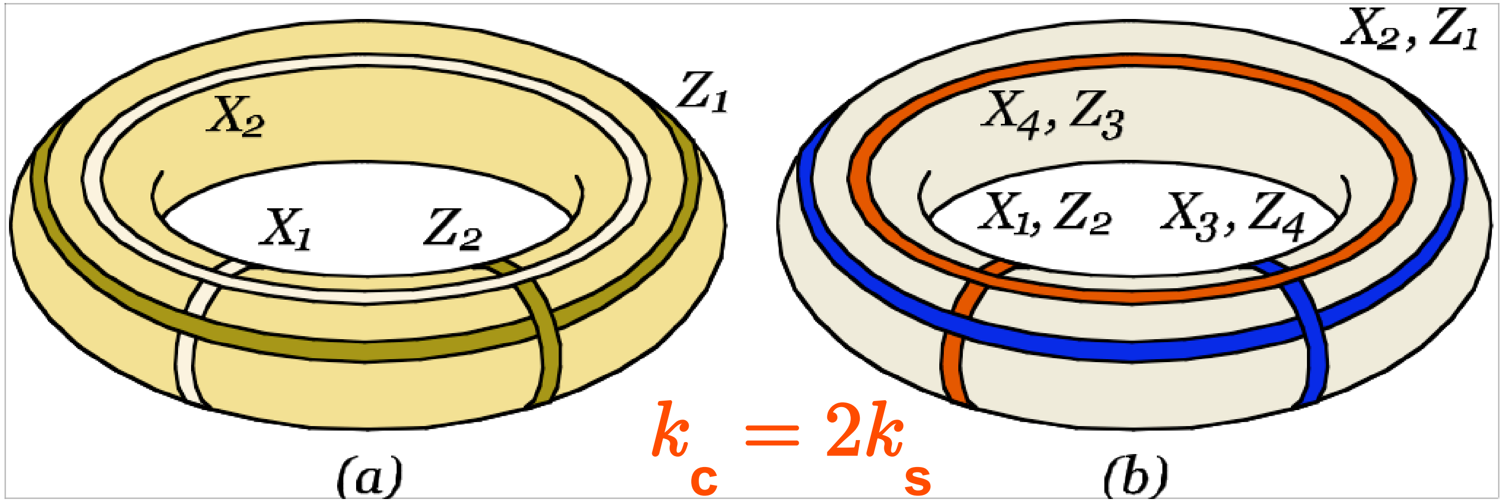
$$S^{x}_{C}\,(S^{z}_{A}\,S^{x}_{B}\,|\mathrm{GS}\rangle) = -S^{z}_{A}\,S^{x}_{B}\,|\mathrm{GS}\rangle$$

- Note that excitations, or their braiding, **play no role** in our computational model. All the operations are carried out in the ground state of the system.

IV. 2-Colexes

Topological 2D Stabilizer Codes: Comparative Study

Pauli operator bases in the torus



(a) $k_c = 2k_s$ (b)

- A color code encodes twice as much logical qubits as a surface code does

•We compute the topological error correcting rate $C := n/d^2$ for surface codes C_s

and color codes C_c
in several instances.

IV. 2-Colexes

Examples of regular codes in the torus with distance d=4

Typical values

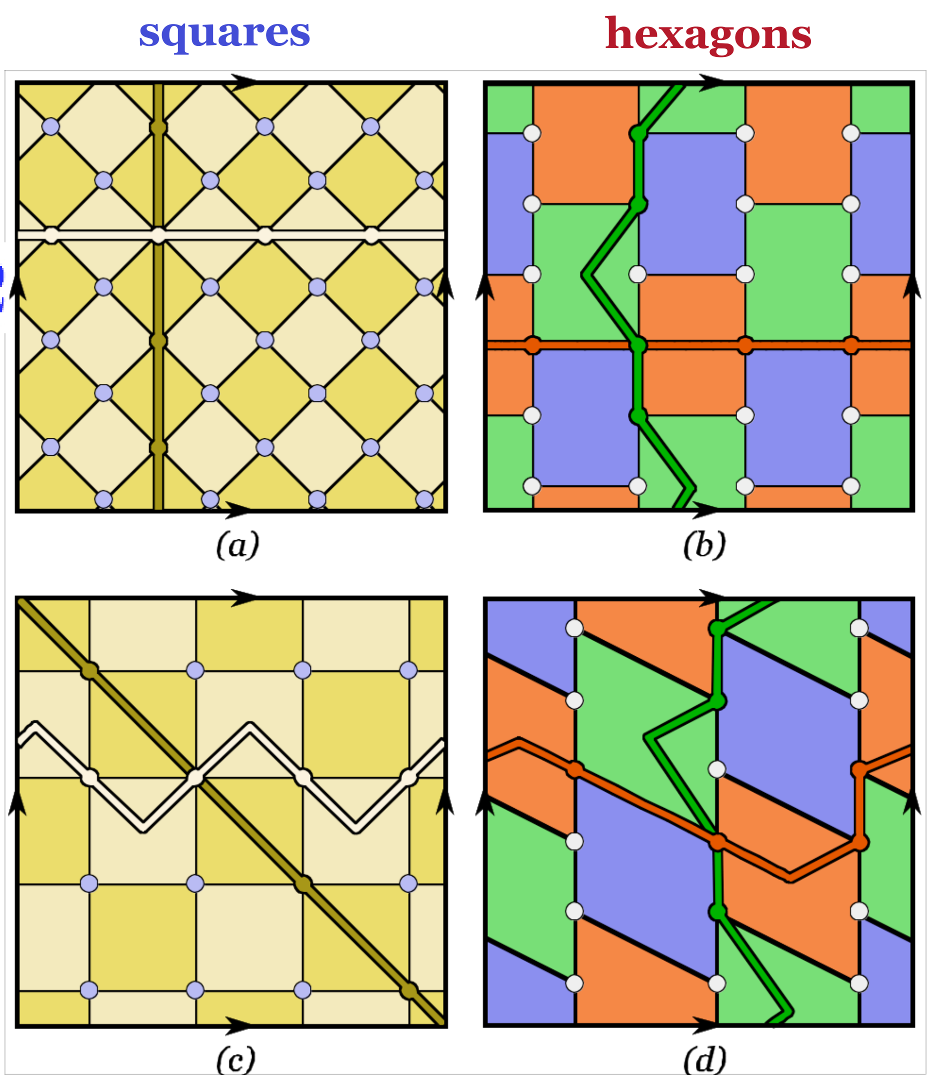
n=32

$$C_s := n_s/d_s^2 = 2$$

Optimal values

n=16

$$C_s^{op} = 1$$



(a) (b) (c) (d)

Typical values

n=24

$$C_c := n_c/d_c^2 = \frac{3}{2}$$

Optimal values

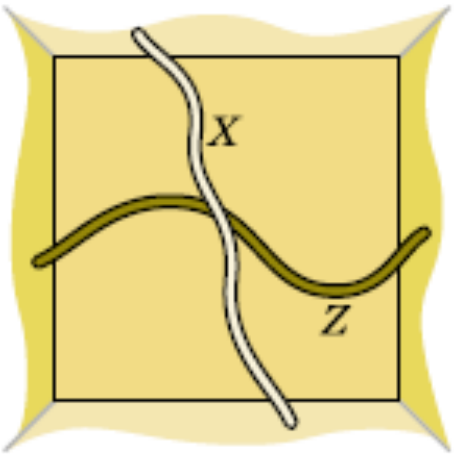
n=18

$$C_c^{op} = \frac{9}{8}$$

IV. 2-Colexes

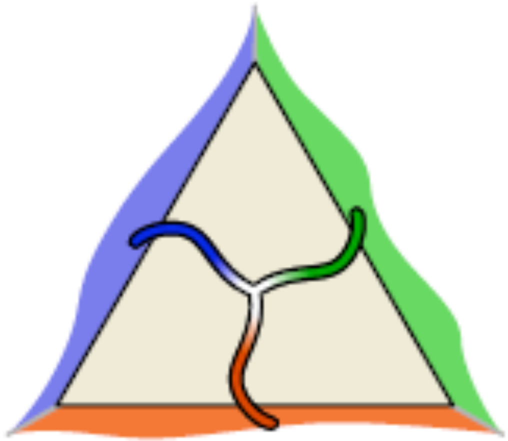
Examples of Planar Codes encoding a single qubit

The colors in the borders represent the class of the missing face



(a)

Kitaev's code



(b)

Color code

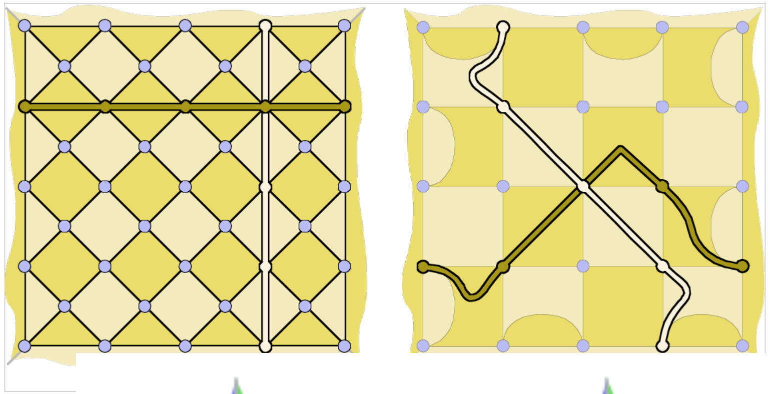
IV. 2-Colexes

Examples of Planar Codes encoding a single qubit

Typical values

n=41, d=5

$$C_s = 2$$

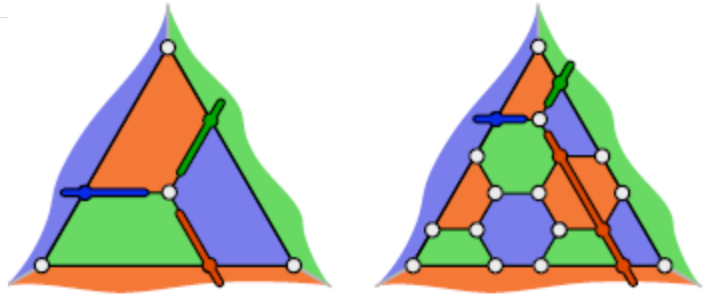


Optimal values

n=25, d=5

$$C^{op}_s = 1$$

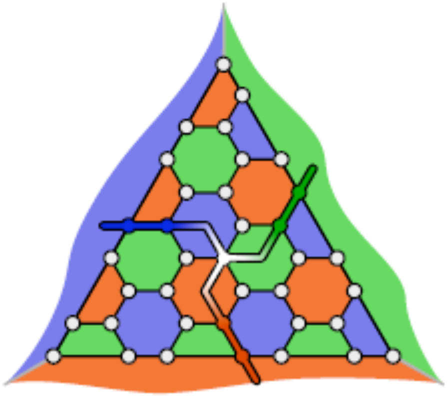
n=7, d=3



(a) (b)

n=19, d=5

$$C^{op}_s = \frac{3}{8}$$

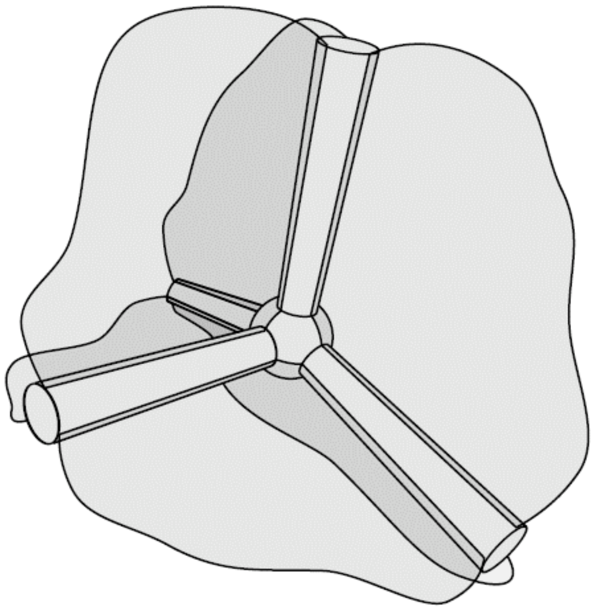


(c)

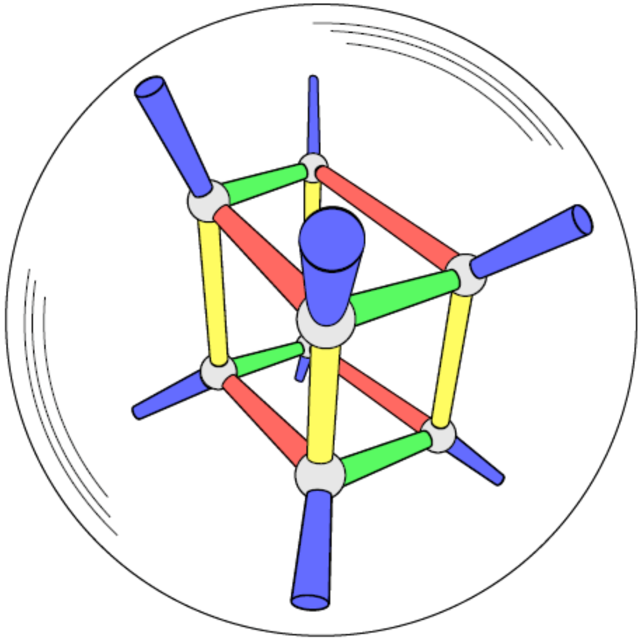
n=37, d=7

V. 3-Colexes

- **3-colexes** are tetravalent lattices with a particular local appearance such that their 3-cells can be 4-colored. They can be built in any compact **3-manifold** without boundary.
- Edges can be colored accordingly, as in the 2-D case.



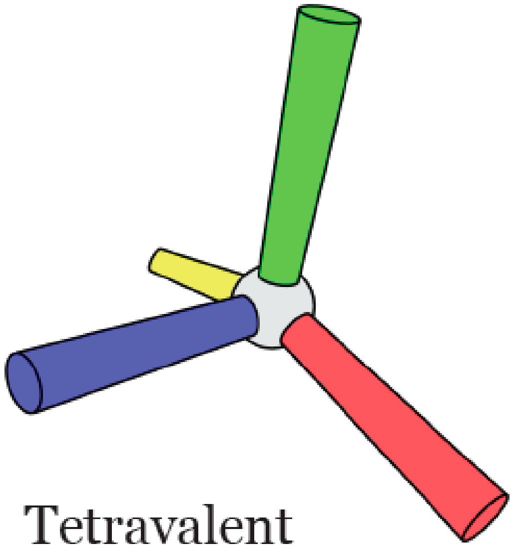
The neigborhood of a vertex.



The simplest 3-colex in the projective space.

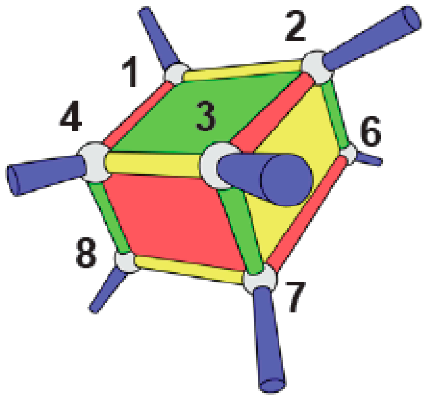
V. 3-Colexes

- **3-Colexes** can be built in any closed 3-manifold:

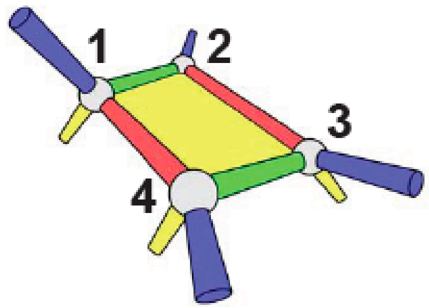


Tetravalent lattice with 4-colored links

A b-cell



A by-face separates b- and y-cells.



Cell operators

$$B_c^X = \bigotimes_{i=1}^{8} X_i$$

Face operators

$$B_f^Z = \bigotimes_{i=1}^{8} Z_i$$

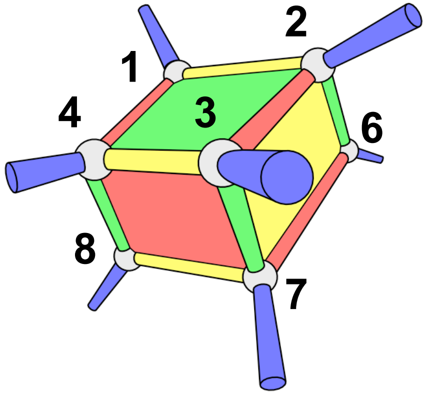
$$H = -\sum_c B_c^X - \sum_f B_f^Z$$

# Encoded qubits = 3h1
h1 = first Betti number

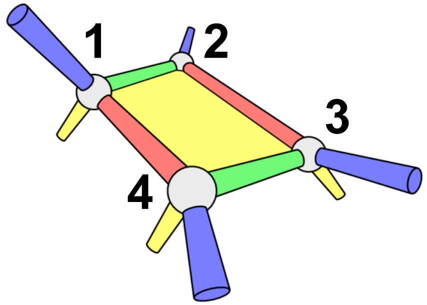
V. 3-Colexes

- This time the generators of S are face and (3-) cell operators.

A b-cell



A by-face separates b- and y-cells.



Cell operators

$$B_c^X = \bigotimes_{i=1}^{8} X_i$$

Face operators

$$B_f^Z = \bigotimes_{i=1}^{8} Z_i$$

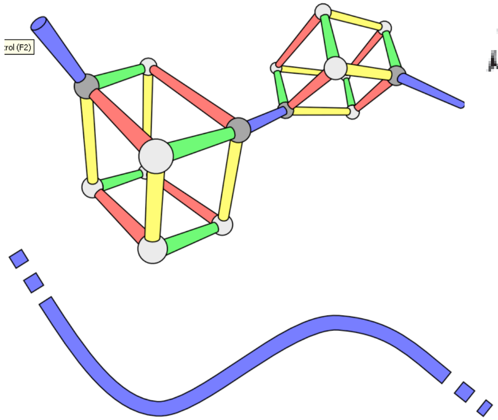
- Therefore there are two different homology groups in the picture, those for 1-chains and for 2-chains. But in fact, due to Poincaré's duality they are the same.

V. 3-Colexes

- Strings are constructed as in 2-D, but now come in four colors. Branching is again possible.
- The new feature are membranes. They come in 6 color combinations and also have branching properties.

String operators

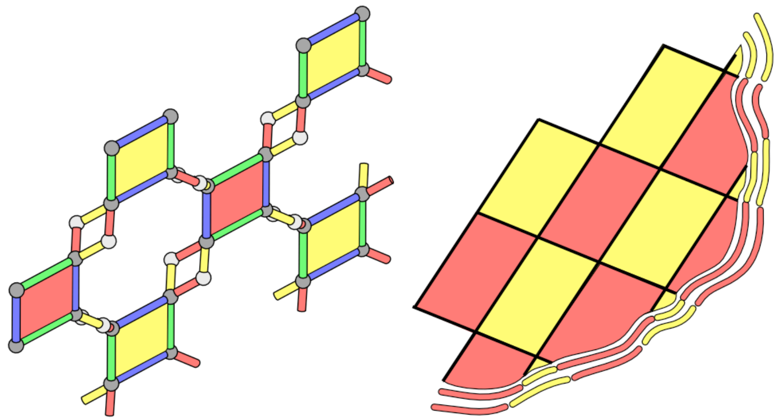
Membrane operators



b-string

$$S^Z = \bigotimes_{\text{string}} Z_i$$

$$M^X = \bigotimes_{\text{membrane}} X_i$$

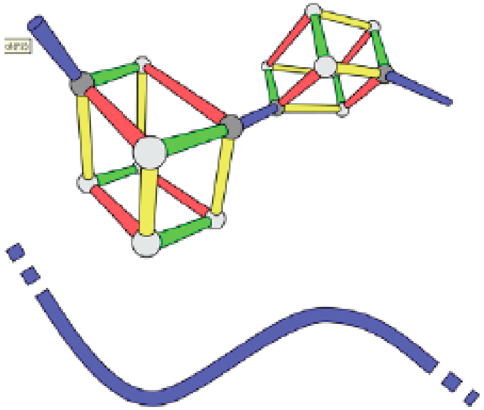


ry-membrane

- There exist appropiate shunk complexes both for strings and for membranes.

V. 3-Colexes

String operators

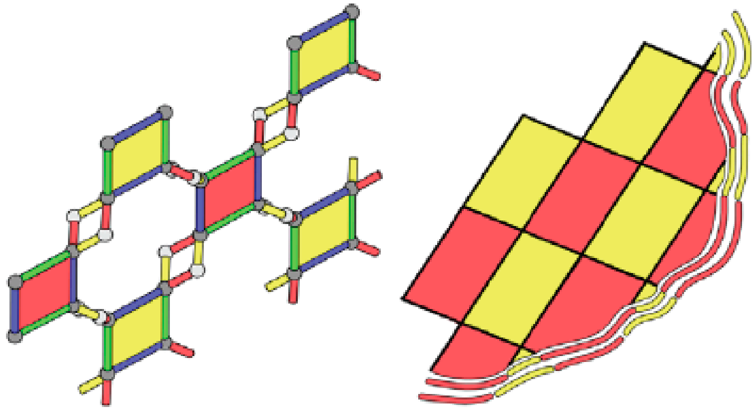


b-string

$$S^Z = \bigotimes_{\text{string}} Z_i$$

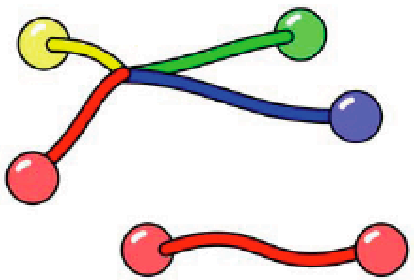
$$M^X = \bigotimes_{\text{membrane}} X_i$$

Membrane operators

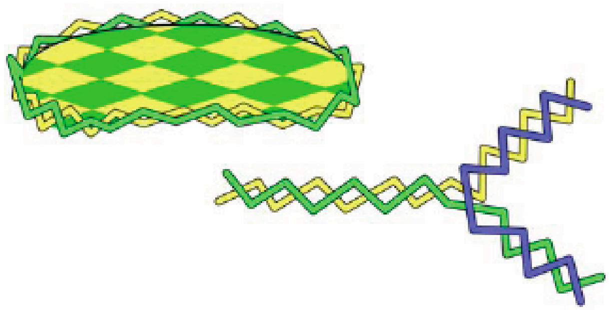


ry-membrane

Excitations



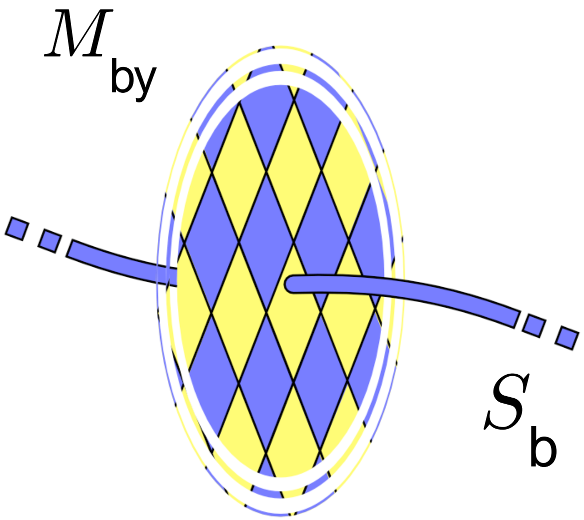
Quasiparticles are the endpoints of strings.



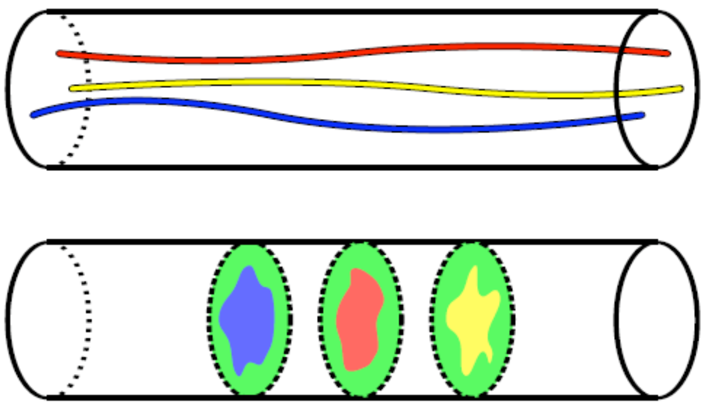
Fluxes are the border of membranes. They can branch.

V. 3-Colexes

- Now there are 3 indendent colors for strings (and similarly 3 color combinations for membranes). Therefore, we expect that the number of encoded qubits will be $3h_1 = 3h_2$
- String and membrane operators always commute, unless they share a color and the string crosses an odd number of times the membrane.



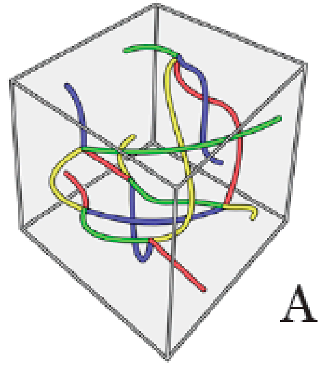
$$\{S^z_b, M^x_{by}\} = 0$$



A Pauli basis for the operators on the 3 qubits encoded in $S^2 x S^1$?

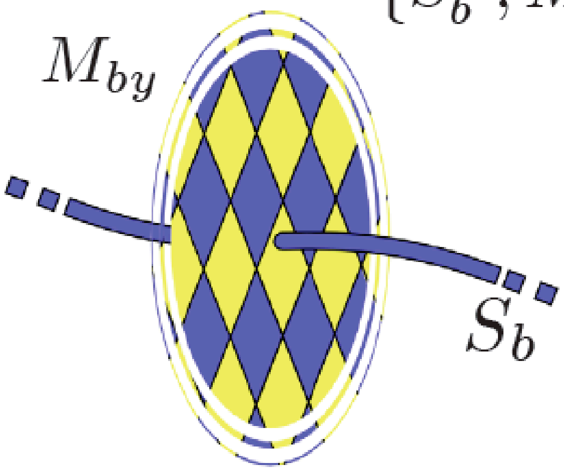
V. 3-Colexes

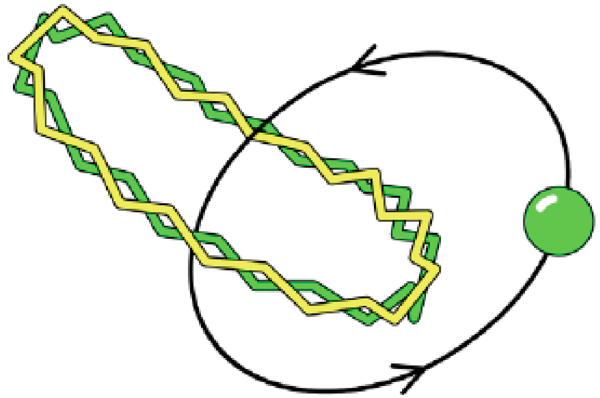
- This system shows a topological order with **string-net and membrane-net** condensation.
- Crossing string and membrane operators with a shared color **anticommute**:



A string-net

$$\{S_b^Z, M_{by}^X\} = 0$$





- If a green quasiparticle **winds** around a green flux, for example, the system gets a global **minus sign**.

D-Colexes

- Higher dimensional **D-Colexes** can also be considered.
- For $D>3$ different **brane-net condensates** are possible. For any pair (p,q) with $p+q=D$ we have a Hamiltonian

$$H_{p,q} = -\sum_{c\in C_{p+1}} B_c^Z - \sum_{c\in C_{q+1}} B_c^X$$

in wich $(p+1)$-cell and $(q+1)$-cell operators are the stabilizers.

- The **degeneracy** of the GS is 2^k with

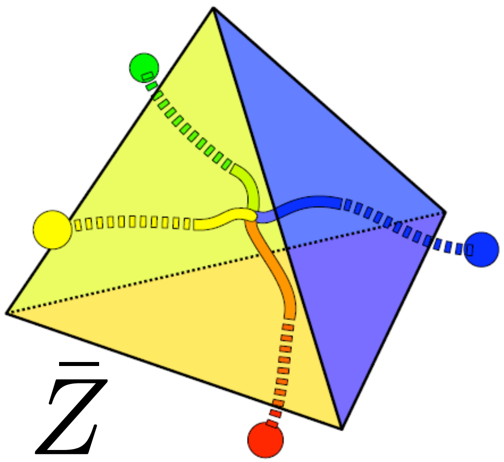
$$k = \binom{D}{p} h_p = \binom{D}{q} h_q$$

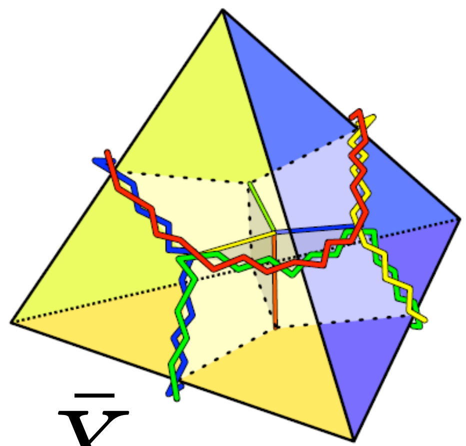
h_s= s-th Betti number

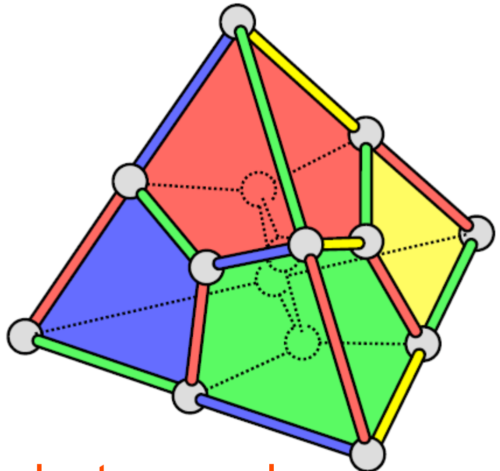
- **Excitations** are extended objects of $p-1$ and $q-1$ dimensions.
- We can braid these excitations and get a global sign. So we talk about **branyons**, for brane-like anyons.

V. 3-Colexes

- 3-Colexes cannot have a practical interest unless we allow boundaries. But this is just a matter of erasing cells. As in two dimensions, boundaries have the color of the erased cell.
- The analogue of triangular codes are **tetrahedral codes**, obtained by erasing a vertex from a 3-sphere.


$\bar{Z}$


$\bar{X}$

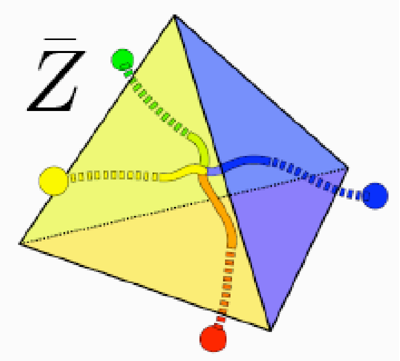

Simplest example

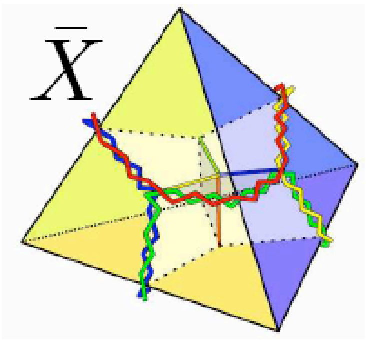
- The desired transversal ***K***$^{1/2}$ gate can be implemented as long as faces have 4x vertices and cells 8x vertices.

V. 3-Colexes

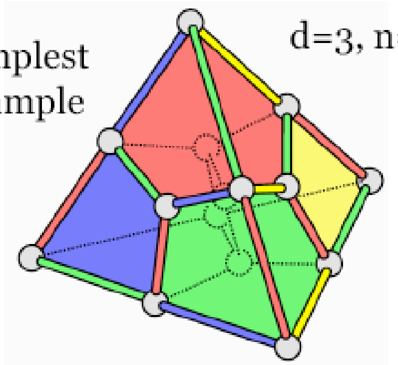
Tetrahedral Codes

- 3-colexes cannot be constructed in our everyday 3D world keeping the locality structure unless we allow boundaries.
- As in *2D*, **borders** are big erased cells and they have the **color** of the **erased cell**.
- Given a border of color *c*, strings can end at it if they are *c*-strings and membranes can end at it if they are *xy*-strings with *x* and *y* different of *c*.
- The analogue of triangular codes are **tetrahedral** codes, which encode a **single** qubit.





Simplest example

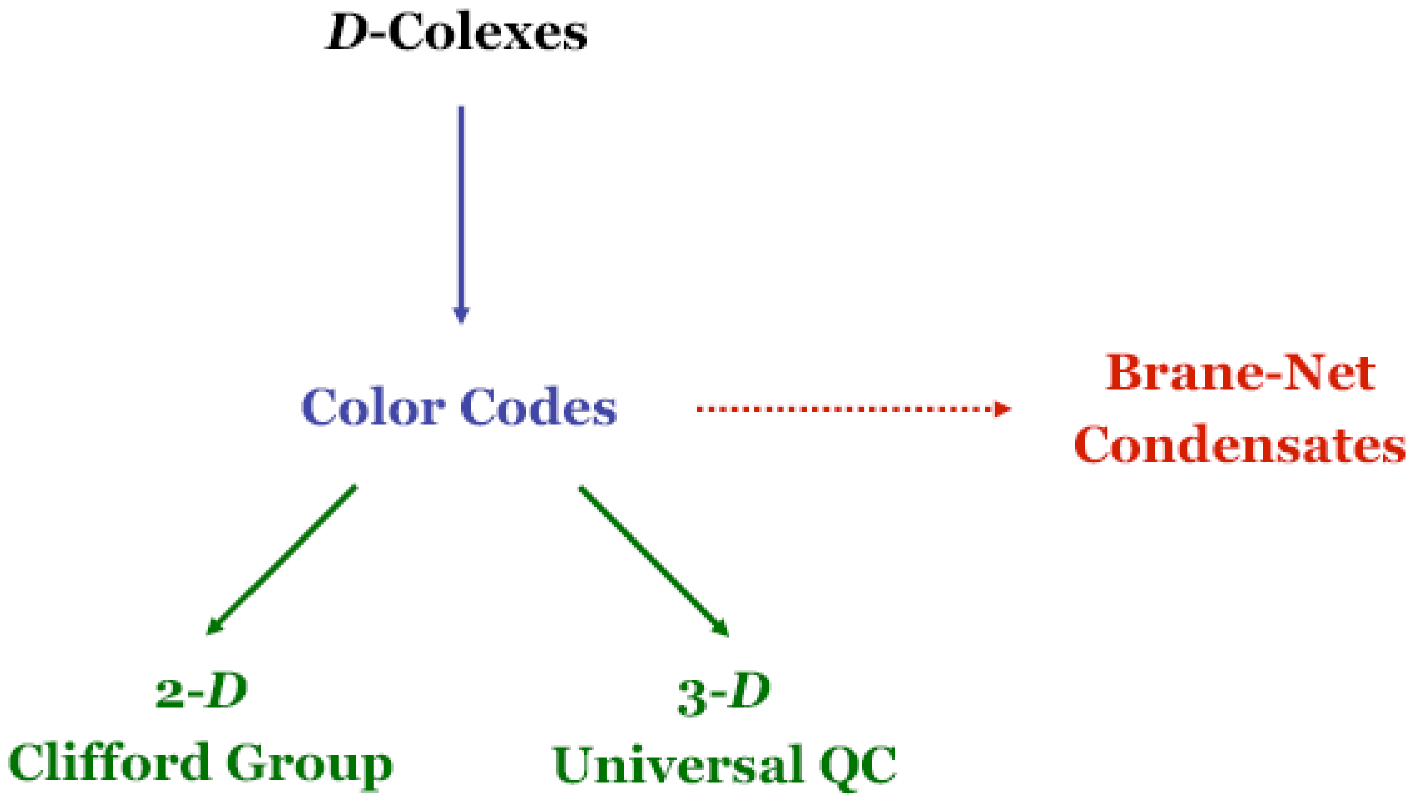


- The desired **transversal** ***K*** gate can be implemented as long as faces have 4x vertices and cells 8x vertices. The trick is analogous to that in Reed-Muller codes:

$$|\hat{0}\rangle := \prod_c (1 + B_c^X)|\mathbf{0}\rangle = \sum_{\mathbf{v}\in V} |\mathbf{v}\rangle \qquad |\hat{1}\rangle := \hat{X}|\hat{0}\rangle \qquad \hat{K}^{1/2}|\hat{0}\rangle = |\hat{0}\rangle$$

$$\forall\, \mathbf{v} \in V \quad \mathrm{wt}(\mathbf{v}) \equiv 0 \mod 8 \qquad l = 1, 3, 5, 7 \qquad \hat{K}^{1/2}|\hat{1}\rangle = i^{l/2}|\hat{1}\rangle$$

Summary


D-Colexes
Color Codes
Brane-Net Condensates
2-*D* Clifford Group
3-*D* Universal QC

Conclusions

- *D*-colexes are *D*-valent complexes with certain coloring properties.
- Topological color codes are obtained from colexes. They have a richer structure than surface codes.
- 2-colexes allow the transversal implementation of Clifford operations.
- 3-colexes allow the transversal implementation of the same gates as Reed-Muller codes.

Conclusions

• There does not exist a fully or complete topological order in D=3 dimensions, unlike in D=2.

• There does not exist a topological order that can discriminate among all the possible topologies in three dimensional manifolds.

• We may introduce the notion of a Topologically Complete (TC) class of quantum Hamiltonians

• We have found a class of topological orders based on the construction of certain lattices called colexes that can distinguish between 3D-manifolds with different homology properties = Homologically Complete (HC) class of quantum Hamiltonians.

• We could envisage the possibility of finding a quantum lattice Hamiltonian, possibly with a non-abelian lattice gauge theory, that could distinguish between any topology in three dimensions by means of its ground state degeneracy.
This would amount to solving the Poincaré conjecture with quantum mechanics.

References

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