Relations between quantum entanglement, tomography and wavelet

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***Abstract*— The intimate connection between the Banach space wavelet reconstruction method for each unitary representation of a given group and homogenous space studied in the last work and quantum entanglement description using group theory considered. We present, universal description of quantum entanglement using group theory and non-commutative characteristic functions for homogenous space and projective representation of group on Banach spaces for some of well known examples, such as: Moyal representation for a spin, Dihedral group and permutation group.**

Keywords: Quantum Entanglement, Separability criteria, Wavelets, Banach Space, Homogenous Space, Projective Group Representations

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I. INTRODUCTION

Entanglement is one of the most fascinating features of quantum mechanics. As Einstein, Podolsky and Rosen [1] pointed out, the quantum states of two physically separated systems that interacted in the past can defy our intuitions about the outcome of local measurements. Moreover, it has recently been recognized that entanglement is a very important resource in quantum information processing[2]. A bipartite mixed state is said to be separable [3] (not entangled) if considered as a convex combination of pure product states.

A separability criterion is based on a simple property that can be shown to hold for every separable state. If some state does not satisfy this property, then it must be entangled. But the converse does not necessarily imply the state to be separable. One of the first and most widely used related criterion is the Positive Partial Transpose (PPT) criterion, introduced by Peres [4]. Furthermore, the necessary and sufficient condition for separability in $H_2 \otimes H_2$ and $H_2 \otimes H_3$ was shown by Horodeckis [5], which was based on a previous work by Woronowicz [6]. However, in higher dimensions, there are PPT states that are nonetheless entangled, as was first shown in [7], based on [6]. These states are called bound entangled states because they have the peculiar property that no entanglement can be distilled from them by local operations [8], [10]. Another approach to distinguish separable states from entangled states involves the so called entanglement witness (EW) [9], [10]. Some of entanglement measures and best separable state conditions using semidefinite programming method is given in [12], [13], [14], [15]. However, no investigation of the separability problem has been carried out there, as the work of Gu predates the seminal paper of Werner[3]. On the other side, the method of characteristic functions has already been successfully applied for studying other then entanglement genuine quantum features of quantum states in the works on non-classical states of quantum harmonic oscillator[16]. Korbicz and Lewenstein by choosing a compact group G and the set of its irreducible, unitary representation as the main ingredients of the mathematical representation of the state space, can define characteristic functions which are applied for testing states' entanglements. Although they do not present any new entanglement test, their results offer a new point of view on the separability problem. Moreover, they were able to translate the positivity of partial transpose (PPT) criterion[4] into the group theoretical language[17]. On the other hand group theoretical approach to quantum entanglement and tomography with wavelet transform has been obtained by some authors [18].

A general framework is already presented for the unification of the Hilbert space wavelets transformation on the one hand, and quasi-distributions and tomographic transformation associated with a given pure quantum states on the other hand [19]. Here in this manuscript we are trying to establish the intimate connection between the quantum entanglement using group theory and non-commutative characteristic functions for homogenous space and projective representation of group on Banach spaces for some of well known examples, such as: Moyal representation for a spin, Dihedral group and permutation group, all which can be represented by density matrices. For density matrices, one defines the norm as $tr()$ this implies the absence of a scalar product in the density matrix space (so it is not a Hilbert space but a Banach space) [24], [25]. Therefore, it is natural to do quantum tomography of any density matrix by using the wavelet transform and its inverse in Banach space connected with the corresponding group representation associated with that density matrix. This obtained quantum tomography by Banach space wavelet method for density states is completely consistent with the quantum tomography obtained by other methods.

The paper is organized as follows:

In section-2 we define wavelet transform based on homogeneous space and projective representation spaces on Banach space. In section-3 a brief recapitulation of group theoretical approach to entanglement for irreducible representation of any compact group is studied. In section-4 we study group theoretical approach to entanglement associated with the unitary irreducible representation on homogenous space for Moyal representation of a spin and projective representation of Dihedral and Permutation group by using the Banach space wavelet transform method. The paper ends with a brief conclusion.

II. Wavelet transform in Banach spaces on homogeneous space and based on projective representation of group:

Wavelet transform on homogeneous space:

The following is a brief recapitulation of some aspects of the theory of wavelets on homogeneous space. We only mention those concepts that will be needed in the sequel, a more detailed treatment may be found for example in [21], [20]. Let G be locally compact group with left Haar measure $d\mu$ and H be a closed subgroup of G. Let π be a continuous representation of a group and $X = G/H$ homogeneous space.

We could define a representation for homogeneous space $X \times X$ in the space $\mathcal{L}(\mathcal{B})$ of bounded linear operators $\mathcal{B} \to \mathcal{B}$:

$$\pi : X \times X \to \mathcal{L}(\mathcal{L}(\mathcal{B})) : \hat{O} \to U(x_1)\hat{O}U(x_2^{-1}), \qquad (1)$$

where if x_1 is equal to x_2, the representation is called adjoint representation, and, if x_2 is equal to identity operator, the representation is called left representation of homogeneous space.

Let $\mathcal{L}(\mathcal{B})$ be the space of bounded linear operator $\mathcal{B} \to \mathcal{B}$ in Banach space. We will say that $b_0 \in \mathcal{B}$ is a vacuum vector if for all $h \in H_1 \times H_2$ we have $\pi(h)b_0 = \chi(h)b_0$ and also the set of vectors $b_{x_1,x_2} = \pi(x_1, x_2)b_0$ forms a family of coherent states, if there exists a continuous non-zero linear functional $l_0 \in \mathcal{B}^*$ (called test functional) and a vector $b_0 \in \mathcal{B}$ (called vacuum vector) such that

$$C(b_0, b_0^{'}) = \int_X < \pi(x_1^{-1}, x_2^{-1})b_0, l_0 > < \pi(x_1, x_2)b_0^{'}, l_0^{'} > d\mu(x_1, x_2), \qquad (2)$$

is non-zero and finite, which is known as the admissibility relation.

If the subgroup H is non-trivial, one does not need to know wavelet transform on the whole group G, but it should be defined on only the homogeneous space G/H, then the reduced wavelet transform $\mathcal{W}$ to a homogeneous space of function L_2 is defined by a representation π of $G \times G$ on $\mathcal{B}$ and a test functional $l_0 \in \mathcal{B}^*$ such that[21]

$$\mathcal{W} : \mathcal{B} \to L_2(X \times X) : \hat{O} \to \hat{O}(x_1, x_2) = [\mathcal{W}\hat{O}](x_1, x_2)$$
$$=< \pi(x_1^{-1}, x_2^{-1})\hat{O}, l_0 >=< \hat{O}, \pi^*(x_1, x_2)l_0 > \quad \forall x_1, x_2 \in X, \qquad (3)$$

where π^* is dual of π.

Wavelet transform based on projective representation:

Let G be a locally compact group with left Haar measure $d\mu$ and Z(H) be a center of group H. Let U be a continuous representation of the group G and $X = G/Z(H)$ be a central extension. In the last subsection we saw that $\hat{\pi}$ has been lifted to an ordinary representation of $G(\chi)$.

Let $\mathcal{L}(\mathcal{B})$ be the space of bounded linear operator $\mathcal{B} \to \mathcal{B}$ in Banach space. We will say that $b_0 \in \mathcal{B}$ is a vacuum vector if for all $h \in Z(H)$ we have $\hat{\pi}(h)b_0 = \chi(h)b_0$ and also the set of vectors $b_x = \hat{\pi}(x)b_0$ forms a family of coherent states, if there exists a continuous non-zero linear functional $l_0 \in \mathcal{B}^*$ (called test functional) and a vector $b_0 \in \mathcal{B}$ (called vacuum vector) such that

$$C(b_0, b_0^{'}) = \int_X < \hat{\pi}(x^{-1})b_0, l_0 > < \hat{\pi}(x)b_0^{'}, l_0^{'} > d\mu(x), \qquad (4)$$

is non-zero and finite, which is known as the admissibility relation.

If the center $Z(H)$ is non-trivial, one does not need to know wavelet transform on the whole group G, but it should be defined on only the central extension $G/Z(H)$, then the reduced wavelet transform $\mathcal{W}$ to a central extension space of function L_2 is defined by a projective representation $\hat{\pi}$ of $G \times G$ on $\mathcal{B}$ and a test functional $l_0 \in \mathcal{B}^*$ such that[21]

$$\mathcal{W} : \mathcal{B} \to L_2(X) : \hat{O} \to \hat{O}(x) = [\mathcal{W}\hat{O}](x)$$
$$=< \hat{\pi}(x^{-1})\hat{O}, l_0 >=< \hat{O}, \hat{\pi}^*(x)l_0 > \quad \forall x \in X, \qquad (5)$$

where $\hat{\pi}^*$ is dual of $\hat{\pi}$.

III. Quantum Entanglement via Group Theory with Wavelet Transform on Banach Space

Group tomography of a compact group G, with an irreducible unitary representation U acting on separable Hilbert space $\mathcal{H}$, means that, every element of $\mathcal{B}(\mathcal{H})$, the Banach algebra of bounded linear operators acting on $\mathcal{H}$, can be constructed by the set $\{U(g), g \in G\}$ according to formula (6), where the set $\{U(g), g \in G\}$ is known as tomographic set and $\Phi(g) = Tr[U^{\dagger}(g)\hat{O}]$ is sampling set or tomogram set of a given operator $\hat{O}$ [33]. When $\mathcal{H}$ is finite-dimensional, the hypothesis that $\{U(g)\}$ is a tomographic set is sufficient to reconstruct any given operator from the tomographic set by using (6), but the case of $\dim(\mathcal{H}) = \infty$ needs a further condition to make sure that every expression converges and that it can be attributed to a precise mathematical meaning. If O is a trace-class operator on $\mathcal{H}$ and $\{U(g)\}$ is a tomographic set and satisfies (4) then we have

$$\hat{O} = \int d\mu(g) Tr[U^{\dagger}(g)\hat{O}]U(g). \qquad (6)$$

Now we try to obtain the above explained tomography via wavelet transforms in Banach space.

In order to do so, we need choose the tomographic set U(g) as a continuous irreducible representation of the wavelet transformation and the identity operator as a vacuum vector. Therefore, the corresponding wavelet transformation takes the following form:

$$\mathcal{W} : \mathcal{B} \mapsto F(g) : \hat{O} \mapsto \hat{\phi}(g) =< \hat{O}, l_g >$$

$$=< \hat{O}, U(g) l_0 >=< \hat{O} U(g)^\dagger, l_0 >= tr(\hat{O} U(g)^\dagger). \quad (7)$$

Korbicz and Lewenstein proceeded to the reformulation of the separability problem in terms of the group theoretical language[17]. For that, let us assume that ρ is separable, i.e., there exist a decomposition of type $\rho = \sum_i p_i |u_i\rangle\langle u_i| \otimes |v_i\rangle\langle v_i|$. By definition of characteristic function[17] or sampling function $\Phi_\rho(g_1, g_2)$ from wavelet transformation in Banach space with above density matrix for irreducible representation $U(g) := U_1(g_1) \otimes U_2(g_2)$ it obtain that [17]

$$\Phi_\rho(g_1, g_2) = tr(\rho U(g)) = \sum_i p_i K_i(g_1) \eta_i(g_2), \quad (8)$$

where $K_i(g_1) =< u_i \mid U(g_1) u_i >$, $\eta_i(g_2) =< v_i \mid U(g_2) v_i >$.

Now we state the following results that are standard and are derived in reference [17].

Theorem 1. Let G be a compact Kinematical group and π, τ are irreducible representation. A state ρ is separable iff its characteristic function can be written in the form $\Phi_\rho(g_1, g_2) = \sum_i p_i K_i(g_1) \eta_i(g_2)$, where $K_i, \eta_i \in \mathcal{P}_1(G)$ (where $\mathcal{P}_1(G)$ is the space of all normalized positive definite functions on G) and the equality holds almost everywhere w.r.t. the Haar measure dg on $G \times G$.

Theorem 2. Let G be a compact Kinematical group and π, τ are irreducible representations at G and ρ is an arbitrary state in $\mathcal{H}_\pi \otimes \mathcal{H}_\tau$: The condition ($\rho$ is separable) $\Rightarrow \tilde{\phi}_\rho \in \mathcal{P}(G \otimes G)$ leads either to PPT criterion for ρ when $\pi \sim \bar{\pi}$ or is empty otherwise[17].

Where $\bar{\pi}(g) := \pi(g^{-1})$ and $\tilde{\phi}(g_1, g_2) := \phi(g_1^{-1}, g_2)$.

A. *Moyal-type representations for a spin*

For a spin s, in [22] is defined a 'Stratonovich-Weyl' correspondence as a rule which maps each operator $\widehat{O}$ on the $(2s+1)$-dimensional Hilbert space $\mathcal{H}_s$ to a function on the phase space of the classical spin, $\mathcal{S}^2$. A *discrete* Moyal formalism is defined as [23].

$$\hat{\Delta}_{\mathbf{n}} = \widehat{U}_{\mathbf{n}} \hat{\Delta}_{\mathbf{n_z}} \hat{U}_{\mathbf{n}}^\dagger, \quad (9)$$

where $\hat{U}_{\mathbf{n}}$ represents a rotation which maps the vector $\mathbf{n}_z$ to $\mathbf{n}$.
By defining the associated kernel as

$$\hat{\Delta}_{\mathbf{n}} \quad = \quad |s, \mathbf{n}\rangle\langle s, \mathbf{n}| \equiv |\mathbf{n}\rangle\langle \mathbf{n}|, \quad (10)$$

$$\hat{\Delta}^{\mathbf{n}} = \sum_{m=-s}^{s} \Delta^m |m, \mathbf{n}\rangle\langle m, \mathbf{n}|. \quad (11)$$

In the wavelet notation, the Banach space is $(2s+1)^2$-dimensional and group is $SU(2)$, the subgroup is U(1) and measure is $d\mu(n) = \frac{2s+1}{4\pi} d(\mathbf{n})$ and the unitary irreducible representation of group is U_n which is the result of with adjoint representation on the any operators in Banach space:

$$\hat{\pi}(n)\hat{\rho} = \hat{U}_n \hat{\rho} \hat{U}_n^\dagger. \quad (12)$$

Then the wavelet transform in this Banach space with the test functional,

$$l_0(\hat{\rho}) = Tr(\hat{\rho} \sum_m \Delta^m |m, n_z\rangle\langle m, n_z|), \quad (13)$$

is given by:

$$\mathcal{W}\hat{\rho} = \phi(n) =< \pi\hat{(n)}^\dagger \hat{O}, l_0 >$$

$$= Tr(\hat{U_n}^\dagger \hat{\rho} \hat{U_n} \sum_m \Delta^m \mid m, n_z >< m, n_z \mid), \quad (14)$$

then we have:

$$\phi(n) = Tr(\hat{\rho} \hat{U_n} \sum_m \Delta^m \mid m, n_z >< m, n_z \mid \hat{U_n}^\dagger) = Tr(\hat{O} \hat{\Delta}^n).$$

In the wavelet notation for the two partite spin system, the irreducible representation $SU(2) \times SU(2)$ is $\hat{\Pi}(n, n') = \Pi\hat{(n)} \otimes \Pi\hat{(n')}$ and the test functional is defined

$$l_0(\hat{\rho}) = Tr(\hat{\rho} \sum_m \Delta^m |m, n_z\rangle\langle m, n_z| \otimes \sum_{m'} \Delta^{m'} |m', n'_z\rangle\langle m', n'_z|), \quad (15)$$

then the characteristic function is defined as:

$$\phi(n, n') =< \pi(\hat{n, n'})^\dagger \hat{O}, l_0 >=$$

$$Tr(\hat{U_n}^\dagger \otimes \hat{U_{n'}}^\dagger \hat{\rho} \hat{U_n} \otimes \hat{U_{n'}} \sum_m \Delta^m \mid m, n_z >< m, n_z \mid$$

$$\otimes \sum_{m'} \Delta^{m'} \mid m', n'_z >< m', n'_z \mid), \quad (16)$$

then we have:

$$\phi(n, n') = Tr(\hat{\rho} \hat{U_n} \otimes \hat{U_{n'}} \sum_m \Delta^m \mid m, n_z >< m, n_z \mid$$

$$\otimes \sum_{m'} \Delta^{m'} \mid m', n'_z >< m', n'_z \mid \hat{U_n}^\dagger \otimes \hat{U_{n'}}^\dagger) =$$

$$Tr(\hat{\rho} \hat{\Delta}^n \otimes \hat{\Delta}^{n'}). \quad (17)$$

From theorems 1 and 2, ρ is separable iff characteristic function written as (8).

As an example we consider $3 \otimes 3$ representation of $SU(2) \otimes SU(2)$ group. Three dimensional representation of SO(3) group as a rotation U_n is defined [17] as

$$U_1(g_1) = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}$$

$$U_2(g_2) = \begin{pmatrix} \lambda'_{11} & \lambda'_{12} & \lambda'_{13} \\ \lambda'_{21} & \lambda'_{22} & \lambda'_{23} \\ \lambda'_{31} & \lambda'_{32} & \lambda'_{33} \end{pmatrix}, \quad (18)$$

where λ_{ij} and $\lambda'_{ij}, i, j = 1, 2, 3$ are defined by using three Euler angles. The $3 \otimes 3$ un-normalized separable states [10], [11] is defined as

$$\rho_m = \sum_k |\psi_{km}\rangle\langle\psi_{km}| = \sum_l |l\rangle\langle l| \otimes |l+m\rangle\langle l+m|,$$

$$\rho'_m = \sum_k |\psi_{mk}\rangle\langle\psi_{mk}| = \sum_{l,l',k} \omega^{m(l-l')}|l\rangle\langle l'| \otimes |l+k\rangle\langle l'+k|,$$

$$\rho''_n = \sum_k |\psi_{nk,k}\rangle\langle\psi_{nk,k}| = \sum_{l,l',k} \omega^{nk(l-l')}|l\rangle\langle l'|\otimes|l+k\rangle\langle l'+k|, \tag{19}$$

where $n = 0, 1, 2$, $m = 0, 1, 2$. One can show that any convex sum of these states is separable and lie at the boundary of the separable region [10]. We can obtain characteristic function $\Phi(g_1, g_2)$ for any convex sum of above separable states but we consider a particular simple case ρ_0. Then characteristic function $\Phi(g_1, g_2)$ for ρ_0 is obtained as

$$\phi(g_1; g_2) = Tr(\hat{\rho}\hat{U}_n \otimes \hat{U}_{n'} \sum_{m=0}^{2} \Delta^m \mid m >< m \mid$$

$$\otimes \sum_{m'=0}^{2} \Delta^{m'} \mid m' >< m' \mid \hat{U}_n^\dagger \otimes \hat{U}_{n'}^\dagger). \tag{20}$$

By some calculation we have

$$\Phi(g_1, g_2) = \sum_{m=0}^{2} \Delta^m (\sum_{i=1}^{3} \lambda_{im}\lambda_{mi}) \sum_{m'=0}^{2} \Delta^{m'} (\sum_{i=1}^{3} \lambda_{im'}\lambda_{m'i})$$

$$= \sum_{m=0}^{2} \Delta^m \sum_{m'=0}^{2} \Delta^{m'} = 1. \tag{21}$$

By definition of $K_i(g_1) = \langle i|U_1(g_1)|i\rangle = \sum_{j=1}^{3} \lambda_{ji}\lambda_{ij} = 1$ and $\eta_i(g_2) = \langle i|U_2(g_2)|i\rangle = \sum_{j=1}^{3} \lambda_{ji}\lambda_{ij} = 1$. Therefore, characteristic function is rewritten as

$$\Phi(g_1, g_2) = \sum_{i=1}^{3} K_i(g_1)\eta_i(g_2) = 1, \tag{22}$$

which is agreement with Theorem 1 and we show that this state is separable in the similar way one can show that ρ_1 and ρ_2 are separable states.

B. Quantum Entanglement based on projective representation of permutation group

Let us consider projective representations of the symmetric (permutation) groups that have long been known to mathematicians, but received little attention from physicists. Such representations were overlooked in physics much like projective representations of the rotation groups were overlooked in the early days of quantum mechanics. One especially useful presentation of the symmetric group S_n on n elements is given by

$$S_n = \{t_1, ..., t_{n-1} : t_i^2 = 1, (t_j t_{j+1})^3 = 1, t_m t_l = t_l t_m\}, \tag{23}$$

where $1 \leq i \leq n-1, 1 \leq j \leq n-2, m \leq l-2$. Here t_i are transpositions,

$$t_1 = (12), t_2 = (23), ..., t_{n-1} = (n-1 \;\; n). \tag{24}$$

Closely related to S_n is the group $\tilde{S}_n$,

$$\tilde{S}_n = \{z, t'_{1;2}, ..., t'_{n-1;n} \mid z^2 = 1, zt'_{i;i+1} = t'_{i;i+1}z,$$
$$t'^{\,2}_{1;2} = z, (t'_{j;j+1}t'_{j+1;j+2})^3 = z,$$
$$t'_{m;m+1}t'_{l;l+1} = zt'_{l;l+1}t'_{m;m+1}\}, \tag{25}$$

where $1 \leq i \leq n-1, 1 \leq j \leq n-2, m \leq l-2$.

A celebrated theorem of Schur (Schur, 1911 [26]) states the following:

(i) The group $\tilde{S}_n$ has order $2(n!)$.

(ii) The subgroup $\{1, z\}$ is central, and is contained in the commutator subgroup of $\tilde{S}_n$, provided n = 4.

(iii) $\tilde{S}_n/\{1, z\} \simeq S_n$.

(iv) If $n < 4$, then every projective representation of S_n is projectively equivalent to a linear representation.

(v) If $n \leq 4$, then every projective representation of S_n is projectively equivalent to a representation $\hat{\pi}$,

$$\rho(S_n) = \{\hat{\pi}(t_1), ..., \hat{\pi}(t_{n-1}) : \hat{\pi}(t_i)^2 = z,$$
$$(\hat{\pi}(t_j)\hat{\pi}(t_{j+1}))^3 = z, \hat{\pi}(t_m)\hat{\pi}(t_l) = z\hat{\pi}(t_l)\hat{\pi}(t_m)\}, \tag{26}$$

where $1 \leq i \leq n-1, 1 \leq j \leq n-2, m \leq l-2$ and $z = \pm 1$.

In the case $z = +1$, $\hat{\pi}$ is a linear representation of S_n.

The group $\tilde{S}_n$ is called the representation group for S_n.

The most elegant way to construct a projective representation $\hat{\pi}(S_n)$ of S_n is by using the complex Clifford algebra $\text{Cliff}_C(V, g) \equiv \mathcal{C}_n$ associated with the real vector space $V = n\mathcal{R}$,

$$\{\gamma_i, \gamma_j\} = 2g(\gamma_i, \gamma_j) \tag{27}$$

Here $\{i\}_{i=1}^n$ is an orthonormal basis of V with respect to the symmetric bilinear form

$$g(\gamma_i, \gamma_j) = +\delta_{ij}. \tag{28}$$

Clearly, any subspace $\bar{V}$ of $V = nR$ generates a subalgebra $\text{Cliff}_C(\bar{V}, \bar{g})$, where $\bar{g}$ is the restriction of g to $\bar{V} \times \bar{V}$. A particularly interesting case is realized when $\bar{V}$ is

$$\bar{V} = \{\sum_{m=1}^{n} \alpha^m \gamma_m : \sum_{m=1}^{n} \alpha^m = 0\} \tag{29}$$

of codimension one, with the corresponding subalgebra denoted by $\bar{C}_{n-1}$ [27]. If we consider a special basis $\{t'_m\}_{m=1}^{n-1} \subset \bar{V}$ (which is not orthonormal) defined by

$$t'_{m;m+1} = \frac{1}{\sqrt{2}}(\gamma_m + \gamma_{m+1}) \quad m = 1, ...n-1, \tag{30}$$

then the group generated by this basis is isomorphic to $\bar{S}_n$. This can be seen by mapping t_i to t'_i and z to -1, and by noticing that

1) For $m = 1, ..., n-1$:

$$t'^{2}_{m;m+1} = -1; \tag{31}$$

2) For $n-2 \geq j$:

$$(t'_{j;j+1}t'_{j+1;j+2})^3 = -1; \tag{32}$$

3) For $n-1 \geq q > m+1$:

$$t'_{m;m+1}t'_{q;q+1} = -t'_{q;q+1}t'_{m;m+1}, \tag{33}$$

as can be checked by direct calculation. One choice for the matrices is provided by the following construction (Brauer and Weyl, 1935 [28]):

$$\begin{cases} \gamma_{2m-1} = \sigma_3 \otimes ...\sigma_3 \otimes (\sigma_1) \otimes 1... \otimes 1, \\ \gamma_{2m} = \sigma_3 \otimes ...\sigma_3 \otimes (\sigma_2) \otimes 1... \otimes 1, \\ m = 1, 2, 3, ...k, \end{cases} \tag{34}$$

for $n = 2k$. Here σ_1, σ_2 occur in the m-th position, the product involves M factors, and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. If $n = 2k+1$, we first add one more matrix,

$$\gamma_{2k+1} = \sigma_3 \otimes ... \otimes \sigma_3 \quad (k \text{ factors}). \tag{35}$$

An irreducible module of $\bar{C}_{n-1}$ restricts that representation to the irreducible representation of $\tilde{S}_n$, since $\{t'_{i;i+1}\}_{i=1}^{n-1}$ generates $\bar{C}_{n-1}$ as an algebra [27]. The simplest (irreducible) non-trivial projective representations of S_n are already surprisingly intricate and have dimensions which grow exponentially with n. They are intimately related to spinor representations of $SO(n)$[29].

Now we try to obtain the characteristic function via wavelets transform in Banach space based on projective representation of permutation group (spinor representation of permutation group). In order to do so, we need to choose the tomographic set $\{\hat{\pi}_{i_1,...,i_m} = \gamma_1^{i_1}\gamma_2^{i_2}, \gamma_3^{i_3}, ..., \gamma_m^{i_m}$, $i_1, i_2, ..., i_m = \{0,1\}\}$ as a projective representation of the wavelet transformation and the identity operator as a vacuum vector. Therefore, the corresponding wavelet transformation takes the following form:

$$\mathcal{W} : \mathcal{B} \mapsto L_2(G) : \hat{\rho} \mapsto \hat{\rho}(i_1, ..., i_m) =$$
$$< \hat{\rho}, l_{(i_1,...,i_m)} >=< \hat{\rho}, \hat{\pi}_{i_1,...,i_m} l_0 >=< \hat{\rho}\hat{\pi}^{\dagger}_{i_1,...,i_m}, l_0 >$$
$$= Tr(\hat{\rho}\hat{\pi}^{\dagger}_{i_1,...,i_m}). \tag{36}$$

In the wavelet notation for the two partite permutation group, for simplicity we reduce our considerations on the irreducible representation $S_2 \times S_2$ is $\hat{\pi}(g_1, g_2) = \pi_{\hat{i_1}, i_2} \otimes \pi_{\hat{j_1}, j_2}$, with $\pi_{i_1,i_2} = \sigma_1^{i_1}\sigma_2^{i_2}, i_1, i_2 = \{0,1\}$, and the characteristic function is defined as

$$\phi(i_1, i_2; j_1, j_2) =< \hat{\rho}, l_{(i_1,i_2;j_1,j_2)} >=< \hat{\rho}, \hat{\pi}_{i_1,i_2;j_1,j_2} l_0 >, \tag{37}$$

using test functional l_0 via trace function the characteristic function is reduced to

$$\phi(i_1, i_2; j_1, j_2) =< \hat{\rho}\hat{\pi}^{\dagger}_{i_1,i_2;j_1,j_2}, l_0 >= Tr(\hat{\rho}\hat{\pi}^{\dagger}_{i_1,i_2;j_1,j_2}). \tag{38}$$

From theorems 1 and 2, ρ is separable iff characteristic function written as (8). The $2 \otimes 2$ Werner states is defined as

$$\rho_f = \frac{1}{6}((2-f)I + (2f-1)\sum_{i,j} |ij\rangle\langle ji|), \ -1 \leq f \leq 1, \tag{39}$$

where I is the identity operator. The positive semidefiniteness condition for the matrix $\Phi_{\alpha\alpha',\beta\beta'} = \phi(g_\alpha^{-1}g_\beta, g_{\alpha'}^{-1}g_{\beta'})$, where these matrices are completely determined by their first rows and the group multiplication table.

Proposition: A function $\phi \in \mathcal{P}(G \times G)$ is separable iff its matrix Φ can be convexly decomposed as follows

$$\Phi = \sum_i p_i K_i \otimes N_i, \tag{40}$$

where for each i, $K_i, N_i \geq 0$ and are defined by the group multiplication table for k_i, η_i, on the other word ϕ is separable if $\Phi^T \geq 0$ ($\Phi_{\alpha\alpha',\beta\beta'} \geq 0 \Rightarrow \Phi_{\beta\alpha',\alpha\beta'} \geq 0$). Now by calculating Φ matrix for the above Werner state one can show that Φ^T will be positive if $0 \leq f \leq 1$ i.e., the Werner state is separable if $0 \leq f \leq 1$.

C. Quantum Entanglement of Dihedral Group Based on Projective Representation

Here we consider the tomography of dihedral group by using irreducible projective representation of this group.

The dihedral group D_n of order $2n$ defined by [30], [27]

$$D_n =< a, b | a^n = 1, b^2 = 1, bab^{-1} = a^{-1} > . \tag{41}$$

Let ε be a primitive nth root of 1 and let

$$\chi : D_n \times D_n \to c^*, \tag{42}$$

be defined by [31]

$$\chi(a^i, a^j b^k) = 1 \quad and \quad \chi(a^i b, a^j b^k) = \varepsilon^j, \tag{43}$$

for all $i, j \in \{0, 1, 2, ..., n-1\}$ and $k \in \{0, 1\}$.

For n=2m is even, for each $r \in \{1, ..., m-1\}$ put

$$A_r = \begin{pmatrix} \varepsilon^r & 0 \\ 0 & \varepsilon^{-r} \end{pmatrix}, \ A_m = \begin{pmatrix} \varepsilon^m & 0 \\ 0 & \varepsilon^m \end{pmatrix}$$
$$B_r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{44}$$

and for n=2m+1 is odd, for each $r \in \{1, ..., m\}$ put [32] is

$$A_r = \begin{pmatrix} \varepsilon^r & 0 \\ 0 & \varepsilon^{-r} \end{pmatrix}, \ B_r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{45}$$

and let $\hat{\pi}_r : D_n \to GL(2, c)$ be defined by

$$\hat{\pi}(i, j) = \hat{\pi}_r(a^i b^j) = A_r^i B_r^j \ , \ \{i \in \{0, 1, ..., n-1\}, j = \{0, 1\}\}. \tag{46}$$

Now we try to obtain the above explained tomography via wavelets transform in Banach space based on projective representation. In order to do so, we need to choose the tomographic set $\hat{\pi}(i, j)$ as a projective representation of the wavelet transformation and the identity operator as a vacuum vector. Therefore, the corresponding wavelet transformation takes the following form:

$$\mathcal{W} : \mathcal{B} \mapsto L_2(G) : \hat{\rho} \mapsto \hat{\rho}(i, j) =$$
$$< \hat{\rho}, l_{(i,j)} >=< \hat{\rho}, \hat{\pi}(i, j) l_0 >=< \hat{\rho}\hat{\pi}^{\dagger}(i, j), l_0 >= Tr(\hat{\rho}\hat{\pi}^{\dagger}(i, j)). \tag{47}$$

In the wavelet notation for the two partite permutation group, for simplicity we reduce our considerations on the irreducible representation $D_2 \times D_2$ is $\hat{\pi}(g_1, g_2) = \pi_{\hat{i_1}, i_2} \otimes \pi_{\hat{j_1}, j_2}$, with $\pi_{i_1,i_2} = A_r^{i_1} B_r^{i_2}, i_1, i_2 = \{0, 1\}$, and the characteristic function is defined as

$$\phi(i_1, i_2; j_1, j_2) =< \hat{\rho}, l_{(i_1,i_2;j_1,j_2)} >=< \hat{\rho}, \hat{\pi}_{i_1,i_2;j_1,j_2} l_0 >, \tag{48}$$

using test functional l_0 via trace function the characteristic function is reduced to

$$\phi(i_1, i_2; j_1, j_2) =< \hat{\rho}\hat{\pi}^{\dagger}_{i_1,i_2;j_1,j_2}, l_0 >= Tr(\hat{\rho}\hat{\pi}^{\dagger}_{i_1,i_2;j_1,j_2}). \tag{49}$$

Let us consider the $2 \otimes 2$ Werner states same as the permutation group

$$\rho_f = \frac{1}{6}((2-f)I + (2f-1)\sum_{i,j} |ij\rangle\langle ji|), \ -1 \leq f \leq 1, \tag{50}$$

where I is the identity operator. The positive semidefiniteness condition for the matrix $\Phi_{\alpha\alpha',\beta\beta'} = \phi(g_{\alpha}^{-1} g_{\beta}, g_{\alpha'}^{-1} g_{\beta'})$, where these matrices are completely determined by their first rows and the group multiplication table. Now by calculating Φ matrix for the above Werner state one can show that Φ^T will be positive if $0 \leq f \leq 1$ i.e., the Werner state is separable if $0 \leq f \leq 1$.

IV. Conclusions

The universal description of quantum entanglement using group theory and non-commutative characteristic functions for homogenous space and projective representation of group on Banach spaces for some of well known examples, such as: Moyal representation for a spin, Dihedral group and permutation group have been considered. Entanglement consideration for others homogenous spaces and projective representation of groups is under investigation.

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