Bob Coecke
University of Oxford

Kindergarten Quantum Mechanics
— beyond the Hilbert space formalism —
Quantum informatics context of this work

“What is the true origin of quantum algorithmic speed-up?”

“How do quantum and classical information interact?”

“What are the limits of quantum computation?”

What is a convincing model thereof?”
“What are the foundational structures of QIC?”
**Foundational Structures for QIC**  
— FET Open (2006) EC STREP-network —

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Ended 2nd out of 500 submissions for an FP6 open call!
Our approach: rebuild QM from scratch!
Kinds/types of systems:

\[ A, B, C, \ldots \]

- e.g. electron, atom, \( n \) qubits, classical data, \ldots
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\[ A, B, C, \ldots \]

- e.g. electron, atom, \( n \) qubits, classical data, ...

Operations/experiments on systems:

\[ A \xrightarrow{f} A, \ A \xrightarrow{g} B, \ B \xrightarrow{h} C, \ldots \]

- e.g. preparation, acting force field, measurement, ...
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- e.g. preparation, acting force field, measurement, ...

Sequential composition of operations:

\[ A \xrightarrow{h \circ g} C \quad := \quad A \xrightarrow{g} B \xrightarrow{h} C \quad A \xrightarrow{1_A} A \]
Kinds/types of systems:

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- e.g. electron, atom, \( n \) qubits, classical data, ...

Operations/experiments on systems:

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Sequential composition of operations:

\[ A \xrightarrow{h \circ g} C \ := \ A \xrightarrow{g} B \xrightarrow{h} C \quad A \xrightarrow{1_A} A \]

Multiplicity of systems/operations:

\[ A \otimes B \quad A \otimes C \xrightarrow{f \otimes g} B \otimes D \]
+ “obvious” rules governing $\circ-\otimes$ interaction
+ “obvious” rules governing $\circ - \otimes$ interaction

$= \text{tensor category}$
+ “obvious” rules governing $\circ \otimes$ interaction

$= \text{tensor category}$

There are graphical calculi comprising these!
+ “obvious” rules governing $\circ \cdot \bigotimes$ interaction

= tensor category

There are graphical calculi comprising these!

graphical language for $\bigotimes$-categories:

$\bigotimes \sim \text{horizontal} \quad \circ \sim \text{vertical}$
+ “obvious” rules governing $\circ$-$\otimes$ interaction

$=$ tensor category

There are graphical calculi comprising these!

graphical language for $\otimes$-categories:

$\otimes \sim \text{horizontal}$  $\circ \sim \text{vertical}$

provable from categorical axioms

$\iff$

derivable in graphical language
\[ f \quad 1_A \quad g \circ f \quad f \otimes g \quad (f \otimes g) \circ h \]
\( f \quad 1_A \quad g \circ f \quad f \otimes g \quad (f \otimes g) \circ h \)
\[ \psi : \mathbb{I} \to A \quad \pi : A \to \mathbb{I} \quad \pi \circ \psi : \mathbb{I} \to \mathbb{I} \]
\[ \psi : I \to A \quad \pi : A \to I \quad \pi \circ \psi : I \to I \]

\[
\begin{array}{c}
\psi \\
\downarrow_{A}
\end{array}
\quad
\begin{array}{c}
\pi \\
\downarrow_{A}
\end{array}
\quad
\begin{array}{c}
\pi \circ \psi \\
\downarrow_{A}
\end{array}
\]

\[ \psi \circ \pi \circ \psi = \pi \]

\[
| \quad \rangle
\]
\[ \psi : I \rightarrow A \quad \pi : A \rightarrow I \quad \pi \circ \psi : I \rightarrow I \]
\( \psi : I \rightarrow A \) \hspace{1cm} \( \pi : A \rightarrow I \) \hspace{1cm} \( \pi \circ \psi : I \rightarrow I \)
\[ \psi : I \to A \quad \quad \pi : A \to I \quad \quad \pi \circ \psi : I \to I \]

\[
\begin{array}{c}
\triangleleft \\
\bigtriangleup
\end{array}
\]

\[
\begin{array}{c}
\psi \\
\bigtriangleup
\end{array}
\]

\[
\begin{array}{c}
\pi \\
\bigtriangleup
\end{array}
\]

\[
\begin{array}{c}
\pi \circ \psi \\
\bigtriangleup
\end{array}
\]

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\begin{array}{c}
\pi \\
\bigtriangleup
\end{array}
\]

\[
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\psi \\
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\end{array}
\]

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\begin{array}{c}
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\psi
\end{array}
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\begin{array}{c}
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\end{array}
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\[
\begin{array}{c}
\psi \\
A \\
\downarrow \\
\psi
\end{array}
\quad
\begin{array}{c}
\pi \\
A \\
\downarrow \\
\pi
\end{array}
\quad
\begin{array}{c}
\pi \circ \psi \\
A \\
\downarrow \\
\psi
\end{array}
\]

\[ \pi \circ \psi = \pi \]

\[
\begin{array}{c}
\pi \\
A \\
\downarrow \\
\psi
\end{array}
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\[ \psi : I \to A \quad \pi : A \to I \quad \pi \circ \psi : I \to I \]

\[\begin{array}{c}
\triangleleft \quad \triangleright
\end{array}\]
\[ \psi : I \to A \quad \pi : A \to I \quad \pi \circ \psi : I \to I \]

\[ \begin{array}{c}
\text{Diagram 1} \\
\begin{array}{c}
\psi \\
\downarrow^A \\
\end{array}
\end{array} \quad \begin{array}{c}
\pi \\
\downarrow^A \\
\end{array} \quad \pi \circ \psi = \begin{array}{c}
\pi \\
\downarrow^A \\
\psi \\
\end{array} \]
\[ \psi : I \to A \quad \pi : A \to I \quad \pi \circ \psi : I \to I \]
\( f : A \rightarrow B \quad \longleftrightarrow \quad f^\dagger : B \rightarrow A \)
QUANTUM STRUCTURE

Abramsky-Coecke (2004) IEEE-LiCS

Selinger (2007) †-Compact categories and CPMs.
Empirical fact: entangled states exist in nature
Empirical fact: entangled states exist in nature

Quantum structure :=
Bell-states exist + their behaviour
System with quantum structure
System with quantum structure

A pair

\((A, \eta : I \to A \otimes A)\)

such that:

\[
\begin{array}{ccc}
A & \sim & I \otimes A \\
\downarrow^{1_A} & & \downarrow^{\sim} \\
A & \sim & A \otimes I
\end{array}
\]

\[
\begin{array}{ccc}
& & \eta^\dagger \otimes 1_A \\
& \sim & (A \otimes A) \otimes A \\
& \downarrow^{\sim} & \\
& & (A \otimes (A \otimes A))
\end{array}
\]
System with quantum structure
System with quantum structure
System with quantum structure
$f^* = f$
\( f^* = f \)

\( f^* = f^\dagger \)
Graphical representation captures their relations
\[(f^*)^* = f^\dagger\]
In Hilb: $f^* \sim$ transposed & $f_\ast \sim$ conjugated
“Sliding” boxes
“Sliding” boxes
“Decorated” normalization
“Decorated” normalization
“Decorated” normalization
Bipartite projector
Bipartite ket & bra
Bipartite state

⇒ Jamiolkowski isomorphism
Classical data flow?
Classical data flow:

\[ f^* = f^† \]
Classical data flow?
Classical data flow

$\Rightarrow$ Quantum teleportation
Classical data flow?
Classical data flow?
CLASSICAL STRUCTURE


Carboni-Walters (1986) *Cartesian bicategories I.*
quantum data cannot be cloned nor deleted
quantum data **cannot** be cloned nor deleted

classical data **CAN** be cloned and deleted
NON-FEATURE:
quantum data cannot be cloned nor deleted

FEATURE:
classical data CAN be cloned and deleted
NON-FEATURE:
quantum data cannot be cloned nor deleted

FEATURE:
classical data CAN be cloned and deleted

Classical data comes with cloning and deleting:

\( (X, \delta : X \rightarrow X \otimes X, \epsilon : X \rightarrow I) \)
NON-FEATURE:
quantum data cannot be cloned nor deleted

FEATURE:
classical data CAN be cloned and deleted

Classical data comes with cloning and deleting:
System with classical structure
System with classical structure

“Frobenius”
(Carboni-Walters 1987 *Cartesian bicategories I*)

“normalisation”
Classical structure $\Rightarrow$ quantum structure =
Classical structure $\Rightarrow$ quantum structure
In Hilb the Bell-state decomposes as:

\[
\mathbb{C} \xrightarrow{\eta_H :: 1 \mapsto \sum_i |ii\rangle} H \otimes H
\]

\[
\mathbb{C} \xrightarrow{\epsilon_H ^\dagger :: 1 \mapsto \sum_i |i\rangle} H
\]

\[
\mathbb{C} \xrightarrow{\delta_H :: |i\rangle \mapsto |ii\rangle} H \otimes H
\]
In Hilb the Bell-state decomposes as:

\[
\mathbb{C} \xrightarrow{\eta_{\mathcal{H}} \colon 1 \mapsto \sum_i |ii\rangle} \mathcal{H} \otimes \mathcal{H}
\]

\[
\epsilon_{\mathcal{H}}^\dagger \colon 1 \mapsto \sum_i |i\rangle
gives\]

\[
\delta_{\mathcal{H}} \colon |i\rangle \mapsto |ii\rangle
\]

This “refinement” specifies a base!
“What’s inside the box?”
“What’s inside the box?”
Notational convention:
Normalisation theorem: A “connected” network build from $\delta$, $\delta^\dagger$, $\epsilon$, $\epsilon^\dagger$ admits a ‘spider-like’ normal form:

Normalisation theorem: A “connected” network built from $\delta, \delta^\dagger, \epsilon, \epsilon^\dagger$ admits a ‘spider-like’ normal form:

```
X X X X X .......... X
X X X X X ........ X
```

proof $\sim$ “fusion” of dots $\Rightarrow$ graphical rewrite system

All five axioms follow from spider-normal-form.
Summary: refining quantum structure
Summary: refining quantum structure
Summary: refining quantum structure
QUANTUM-CLASSICAL FLOW
Quantum measurement:

\[ \mathcal{M} : A \rightarrow X \otimes A \]
Quantum measurement:

\[ M : A \rightarrow X \otimes A \]
Quantum measurement:

\[ M : A \to X \otimes A \]
Quantum measurement:

\[ \mathcal{M} : A \rightarrow X \otimes A \]

\[ \Rightarrow \text{von Neumann projection postulate.} \]
Quantum measurement:

\[ \mathcal{M} : A \rightarrow X \otimes A \]

\[
\begin{array}{ccc}
A & \rightarrow & X \otimes A \\
| & \downarrow \mathcal{M} & \\
X \otimes A & \rightarrow & X \otimes X \otimes A \\
& | & \\
& \delta \otimes 1_A & \\
\end{array}
\]
Quantum measurement:

\[ \mathcal{M} : A \rightarrow X \otimes A \]
Quantum measurement:

\[
\begin{align*}
A & \xrightarrow{\mathcal{M}} A \\
\mathcal{M} & \\
X \otimes A & \xrightarrow{\lambda_A^\dagger \circ (\epsilon \otimes 1_A)} A
\end{align*}
\]
Quantum measurement:

\[ \mathcal{M} : A \rightarrow X \otimes A \]

\[ \Rightarrow \text{ "indexed" self-adjointness.} \]
Thm. Self-adjoint Eilenberg-Moore coalgebras for $\mathcal{H} \otimes -$ : $\text{FdHilb} \to \text{FdHilb}$ are exactly $\dim \mathcal{H}$-outcome quantum measurements.
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Coalg-square $\Rightarrow$
- idempotence $P_i^2 = P_i$
- mutual orthogonality $P_i \circ P_j \neq i = 0$

Coalg-triangle $\Rightarrow$
- Completeness of spectrum $\sum_i P_i = 1_\mathcal{H}$

Self-adjointness $\Rightarrow$
- Orthogonality of projectors $P_i^\dagger = P_i$
What do these mean?
What do these mean?
What do these mean?

Minimal requirements for reasonable notion of measurement
What do these mean?
What do these mean?

Asserts *no-faster-than-light* communication
ALICE

BOB
Teleportation:

Alice

Bob
Bipartite quantum measurement:
Bipartite quantum measurement:
Bipartite quantum measurement:
Bipartite quantum measurement:
Teleportation enabling measurement:

\[
\begin{align*}
\text{abstracts } & \quad \dim(X) \geq (\dim(A))^2 \\
\text{and } & \quad \Tr(U_x \circ U_y) = \delta_{xy}.
\end{align*}
\]
Teleportation enabling measurement:

\[
\begin{align*}
\text{abstracts } \dim(X) &\geq (\dim(A))^2 \quad \text{and} \quad \text{Tr}(U_x \circ U_y^\dagger) = \delta_{xy}. \\
\text{abstracts unitarity of } \{U_x\}_x \quad \text{i.e.} \quad U_x^\dagger \circ U_x = U_x \circ U_x^\dagger =
\end{align*}
\]
$1_A$. 
Teleportation:
Intended behavior:

Alice

Bob
Proof:
Dense coding:
Intended behavior:
Proof:
Other things we can do:

- proof correctness, generalize and find required structural resources of measurement based schemes’s.
- CPMs, POVMs and Naimark’s extension theorem
- resource inequalities e.g. coherent communication

Ross Duncan’s talk:\(^1\)

- computing with basic quantum gates
- prove universality of one-way computing
- compute the quantum Fourier transform

\(^1\)EXPOSES THE COMPUTATIONAL POWER OF MULTIPLE CLASSICAL CONTEXTS i.e. WE CAN USE CLASSICAL STRUCTURE NOT ONLY FOR CONTROL BUT ALSO FOR “RAW” QUANTUM CALCULUS
Classical species:

Classical map

Relation \((\delta\text{-}lax, \epsilon\text{-}lax)\)

Partial map \((\delta, \epsilon\text{-}lax)\)

Stochastic map \((\epsilon)\)

Total map \((\delta, \epsilon)\)

Bistochastic map \((\epsilon, \epsilon^\dagger)\)

Permutation \((\delta, \epsilon, \delta^\dagger, \epsilon^\dagger)\)
Classical maps are broadcast-able maps

Environment

\[ f = \text{environment} \]