IICQI14

Gaussian Bosonic Channels: conjectures, proofs, and bounds

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NEST SCUOLA NORMALE SUPERIORE PISA

CM condensed matter and quantum information

THE RESULT IN A NUTSHELL

"Pares cum paribus facillime congregantur"

Cicero, De Senectute

"CERTAIN FUNCTIONALS* EVALUATED AT THE OUTPUT OF A BOSONIC GAUSSIAN CHANNEL (BGC) ARE OPTIMIZED (SAY MINIMIZED) BY GAUSSIAN INPUT STATES"

$$\min_{\rho} \mathcal{F}(\Phi(\rho)) = \min_{\rho_G} \mathcal{F}(\Phi(\rho_G))$$

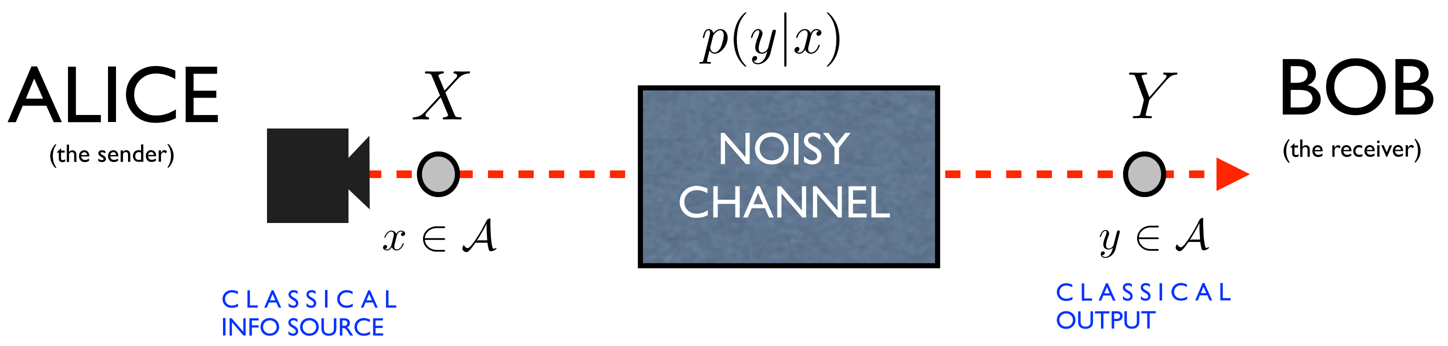
***VON NEUMANN ENTROPY**
RENYI ENTROPIES
CONCAVE FUNCTIONALS
HOLEVO INFORMATION

Outlook

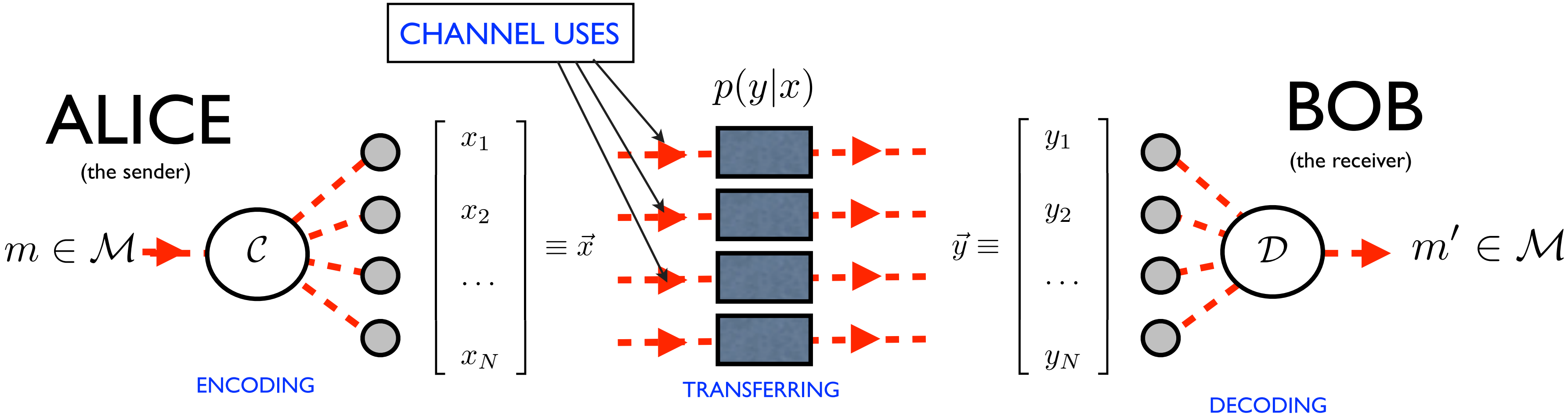
1. Sending classical messages over a quantum channel
2. Bosonic Gaussian Channels (BGCs)
3. "The Conjectures"
4. Solutions
5. Conclusions and Perspectives

1.Sending classical messages over a quantum channel

CLASSICAL COMMUNICATION LINE



CLASSICAL COMMUNICATION LINE



RATE = $R = \frac{\#\text{Bits}}{\#\text{channel uses}} = \frac{\log_2 M}{N}$

$$C = \max_{achievable} R = \lim_{\epsilon \to 0} \limsup_{N \to \infty} \left\{ \frac{\log_2 M}{N} \middle| \exists\, \mathbf{C}_{M,N} \quad \text{such that} \quad P_{err}(\mathbf{C}) < \epsilon \right\}$$

Shannon NOISY CHANNEL CODING THEOREM

$$C = \max_{p(x)} H(X:Y)$$

single letter formula ...
no regularization needed over N

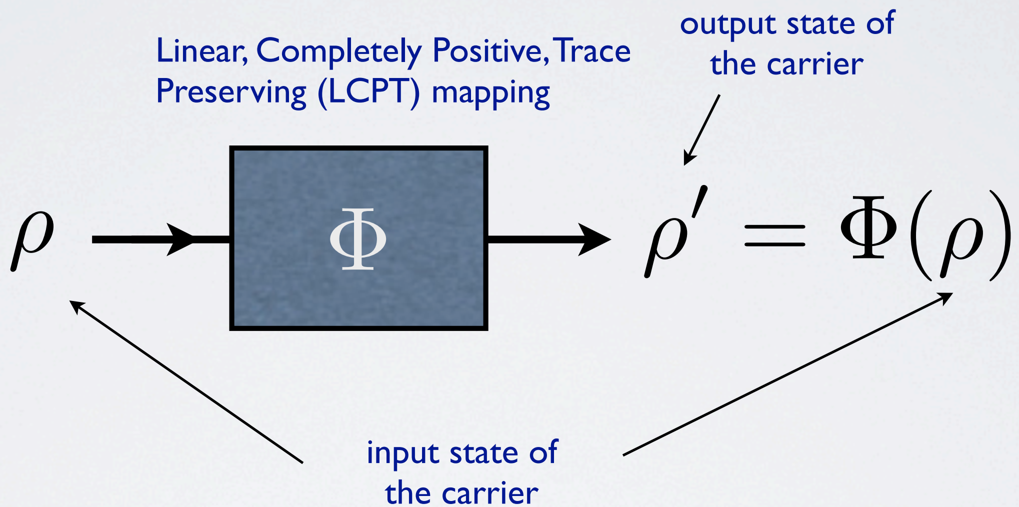
$$H(X:Y) = H(X) + H(Y) - H(X,Y)$$

MUTUAL INFORMATION of X,Y

$$H(X) = -\sum_x p(x) \log_2 p(x)$$

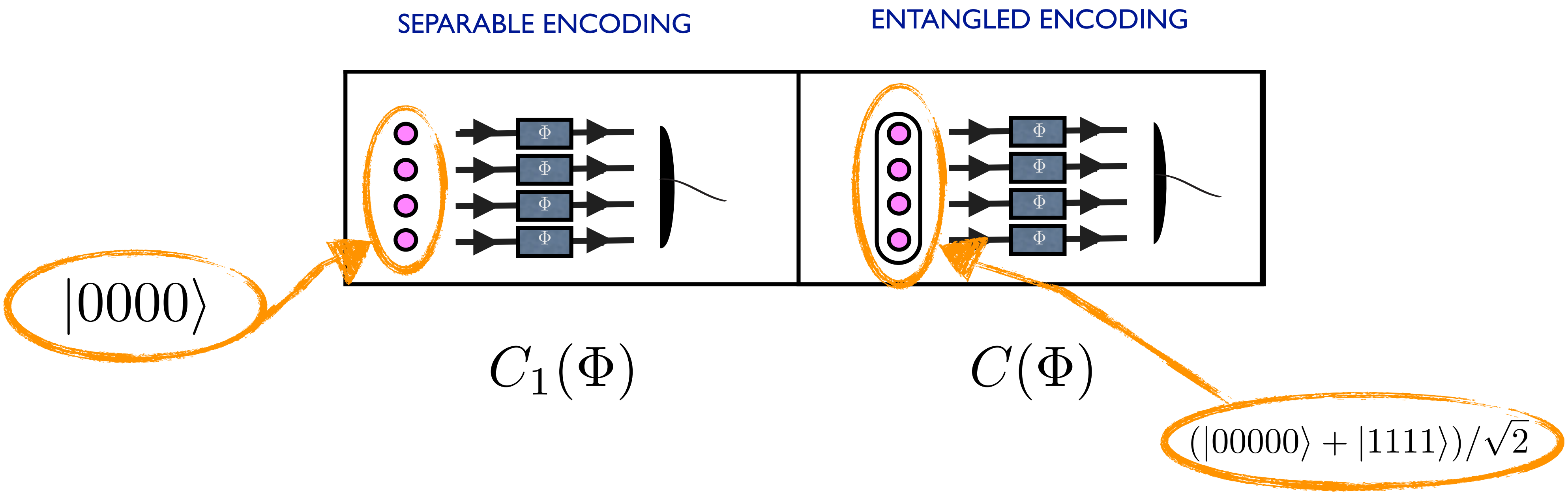
Sending classical messages on a Quantum Channel

INPUT/OUPUT FORMALISM



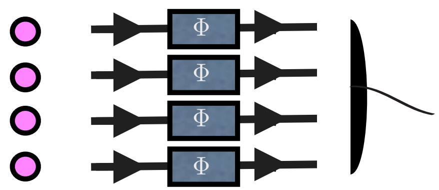
As in the classical theory we can define the CAPACITY of the Channels as:

$$C = \max_{achievable} R = \lim_{\epsilon \to 0} \limsup_{N \to \infty} \left\{ \frac{\log_2 M}{N} \,\middle|\, \exists\, \mathbf{C}_{M,N} \quad \text{such that} \quad P_{err}(\mathbf{C}) < \epsilon \right\}$$



Holevo-Schumacher-Westmoreland (HSW)

CHANNEL CODING THEOREM (I)



if we restrict the ENCODING to only those which produce SEPARABLE (non entangled) CODEWORDS, then

$$C_1(\Phi) \equiv \max_{\mathrm{ENS}} \; C_\chi(\Phi(\textsc{ens}))$$

= HOLEVO CAPACITY OF THE CHANNEL Φ

MAXIMIZED OVER ALL POSSIBLE ENSEMBLES

HOLEVO IEEE 44, 269 (1998)
SCHUMACHER and WESTMORELAND PRA 56, 2629 (1998)

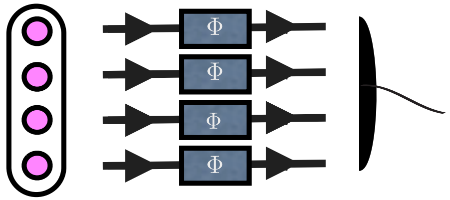
$$\textsc{ens} = \{\rho_j ; p_j \, : \, \rho_j \in \mathfrak{S}(\mathcal{H})\}$$ = ensemble of input states of the information carrier

$$\Phi(\textsc{ens}) = \{\Phi(\rho_j); p_j \, : \, \rho_j \in \mathfrak{S}(\mathcal{H})\}$$ = output ensemble associated with ENS

$$C_\chi(\textsc{ens}) = S(\sum_j p_j \rho_j) - \sum_j p_j S(\rho_j)$$ = HOLEVO INFO of the ensemble ENS

Holevo-Schumacher-Westmoreland (HSW)

CHANNEL CODING THEOREM (II)



if we allows for ANY ENCODING including those which produce ENTANGLED CODEWORDS, then

$$C(\Phi) = \lim_{N\to\infty} \frac{C_1(\Phi^{\otimes N})}{N}$$

REGULARIZATION OVER CHANNEL USES

$$C_1(\Phi^{\otimes N}) = \max_{\text{ENS}} \; C_\chi(\Phi^{\otimes N}(\text{ENS}))$$ = HOLEVO CAPACITY OF THE CHANNEL $\Phi^{\otimes N}$

MAXIMIZED OVER ALL POSSIBLE N-dim ENSEMBLES

$$C(\Phi) \geq C_1(\Phi)$$

$$C(\Phi) \geq C_1(\Phi)$$

ADDITIVITY ISSUE:

C is no longer a single expression formula (we have to take the limit over arbitrarily large *N*).

$$C(\Phi) \stackrel{?}{=} C_1(\Phi)$$

ADDITIVITY Problem of Holevo INFO

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ADDITIVITY Problem of Holevo INFO

SHOR EQUIVALENCE THEOREM (2004):

(i) additivity of the minimum entropy output of a quantum channel

(ii) additivity of entanglement of formation

(iii) strong super-additivity of the entanglement of formation

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FALSE

ADDITIVITY problem of Holevo INFO

SHOR EQUIVALENCE THEOREM (2004):

(i) additivity of the minimum entropy output of a quantum channel FALSE

(ii) additivity of entanglement of formation FALSE

(iii) strong super-additivity of the entanglement of formation FALSE

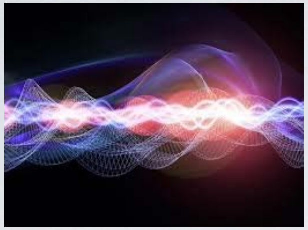
Hastings, Nature Physics 5, 255 (2008)

$$C(\Phi) = \lim_{N\to\infty} \frac{C_1(\Phi^{\otimes N})}{N}$$

still, for some special channels it may be the case that the additivity holds ...

2. Bosonic Gaussian Channels (BGCs)

Bosonic Gaussian Channels (BGCs)

ALICE  BOB

each INPUT SYSTEM is a collection of (say) s independent optical modes

$$\left[a_j\,,a_k^\dagger\right] = \delta_{jk}$$

annihilation operator of the j-th mode

$$D(z) = \exp[\mathbf{a}^\dagger z - z^\dagger \mathbf{a}] = \exp \sum_{j=1}^{s} \left(z_j a_j^\dagger - z_j^* a_j \right)$$

displacement (or Weyl) operators

$z = (z_1, z_2, \cdots, z_s)^t$ $\qquad$ $\mathbf{a} = [a_1, \ldots, a_s]^t$

$$\rho = \frac{1}{\pi^s} \int d^{2s} z \chi(z) D(-z) \quad \Leftrightarrow \quad \chi(z) = \mathrm{Tr}[\rho D(z)]$$

Symmetrically Ordered Characteristic Function

Bosonic Gaussian Channels (BGCs)

A state ρ is a Gaussian state iff $\chi(z)$ is a Gaussian function

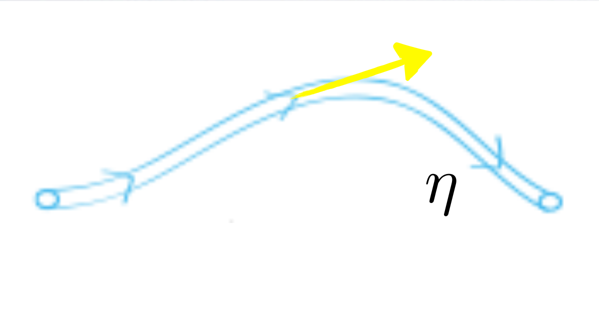
Vacuum state,
Coherent states,
Squeezed states,
Thermal states.

A LCPT map Φ is a BGC if it sends Gaussian input states ρ into output Gaussian states $\Phi(\rho)$

Holevo, Werner PRA 63, 1997

Attenuation (loss), Amplification, Squeezing, Thermalization processes

ALICE



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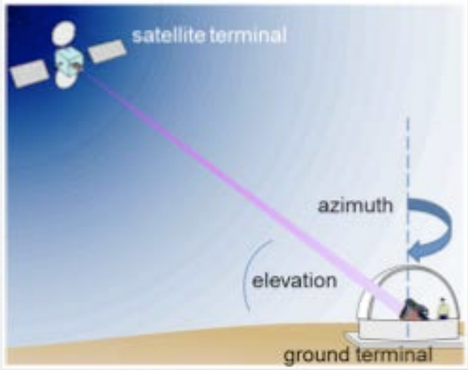


BOB

Caves, Drummond RMP 1994

Bosonic Gaussian Channels (BGCs)

for phase-covariant
channels (multimode channels)

$$\chi(z) = \mathrm{Tr}[\rho D(z)] \quad \rightarrow$$

BOSONIC GAUSSIAN
CHANNEL

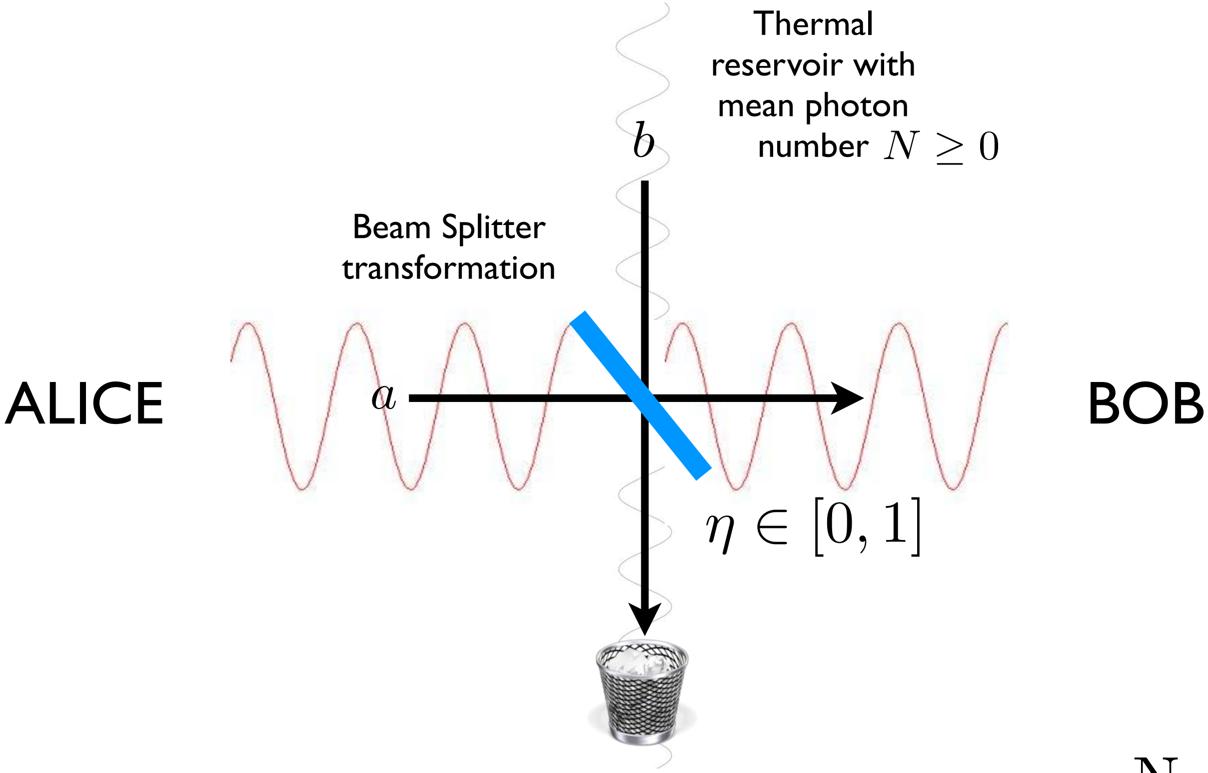
$$\rightarrow \quad \chi'(z) = \mathrm{Tr}[\Phi(\rho) D(z)] = \chi(K^\dagger z) \; \exp[-z^\dagger \mu z]$$

$$\mu \geq \pm \frac{1}{2} \left(I - K K^\dagger \right)$$

Bosonic Gaussian Channels (BGCs)

Attenuator (or thermal) single mode channel (s=1)

$$\chi'(z) = \chi(\sqrt{\eta} z)\, e^{-(1-\eta)(N+1/2)|z|^2}$$



$$\mathcal{E}_\eta^N(\rho) = \mathrm{Tr}_E[U(\rho \otimes \sigma_E)U^\dagger]$$

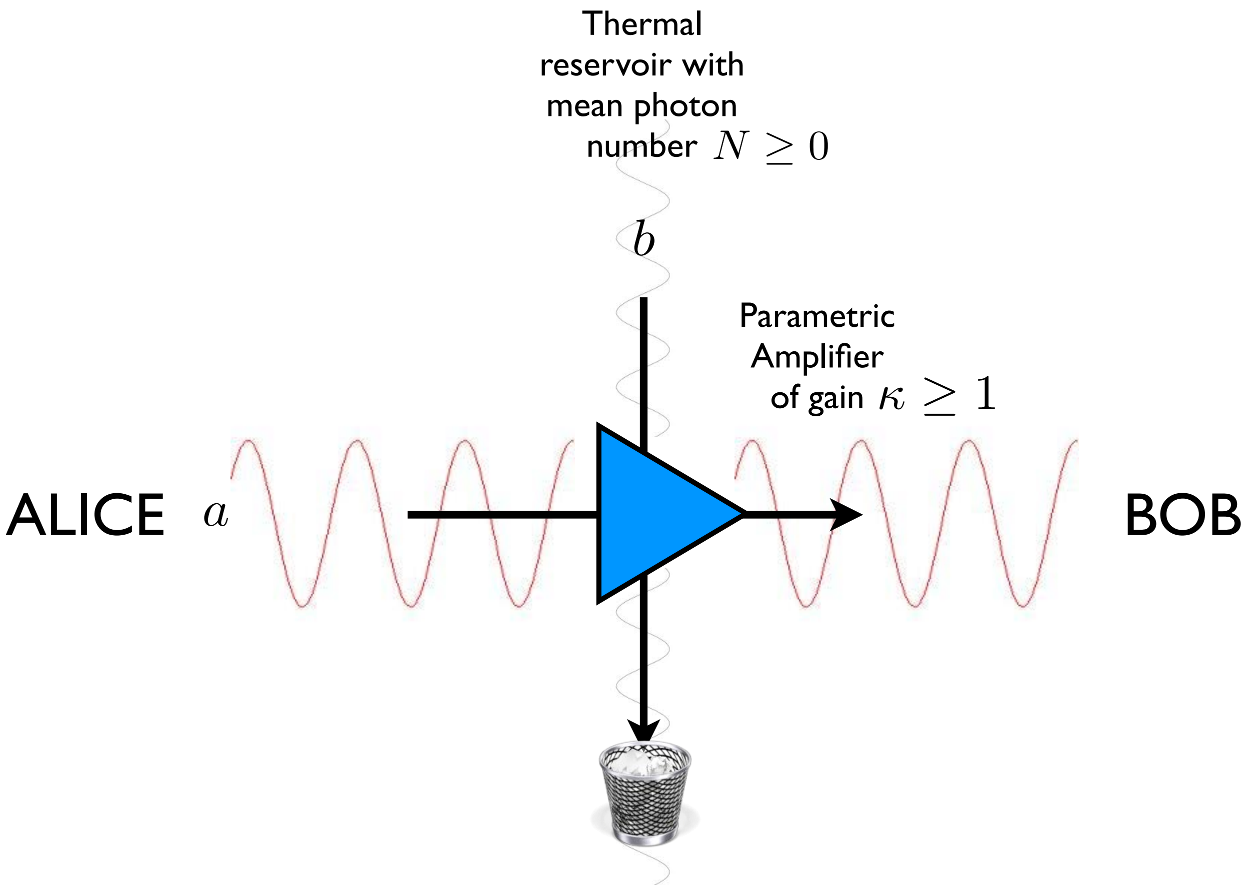
N=0

$$\mathcal{E}_\eta^0(\rho) = \mathrm{Tr}_E[U(\rho \otimes |\emptyset\rangle\langle\emptyset|)U^\dagger]$$

purely lossy channel (minimal noise attenuator)

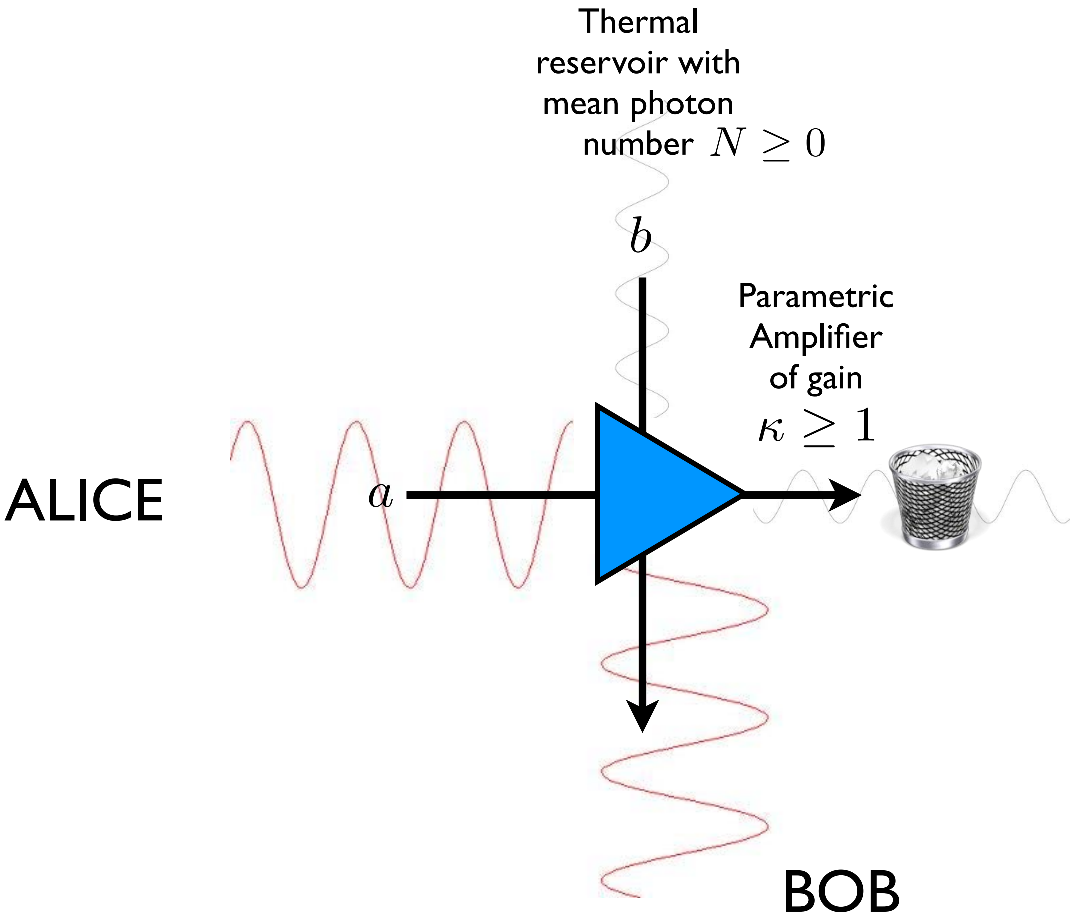
Bosonic Gaussian Channels (BGCs)

Amplifier channel single mode channel (s=1) $\chi'(z) = \chi(\sqrt{\kappa} z)\; e^{-(\kappa-1)(N+1/2)|z|^2}$



$$\mathcal{A}_\kappa^N(\rho) = \mathrm{Tr}_E[U(\rho \otimes \sigma_E)U^\dagger]$$

Bosonic Gaussian Channels (BGCs)



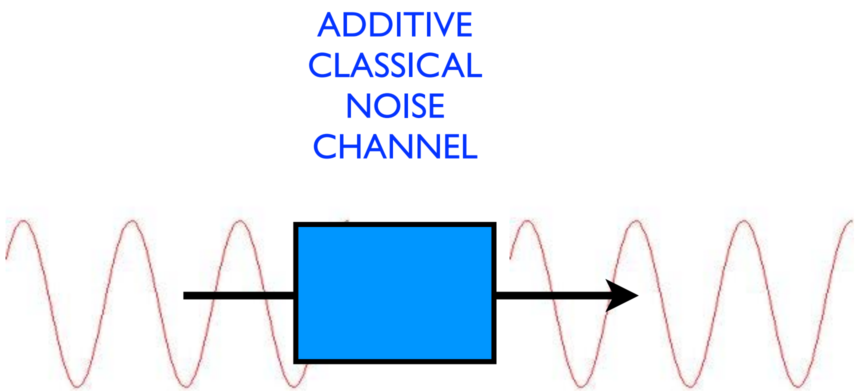
$$\tilde{\mathcal{A}}_{\kappa}^{N}(\rho) = \mathrm{Tr}_S[U(\rho \otimes \sigma_E)U^{\dagger}]$$

weak complementary of an amplifier channel

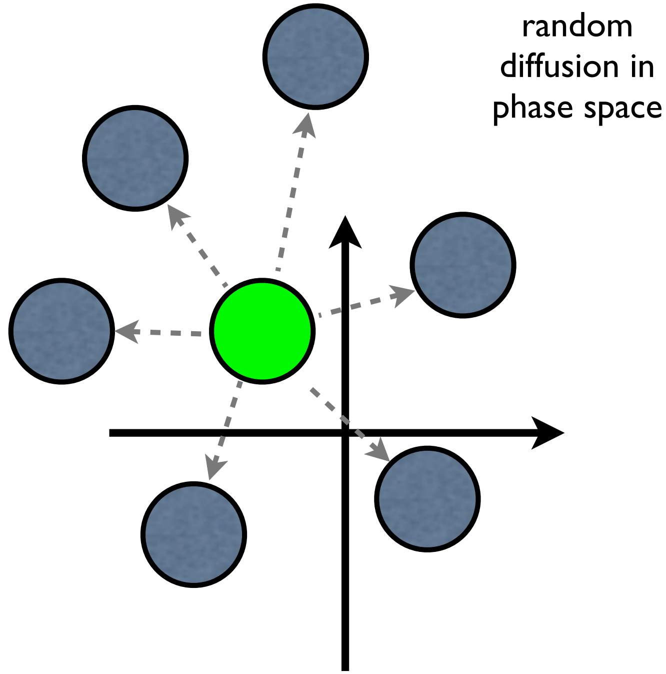
$$\chi'(z) = \chi(-\sqrt{\kappa - 1} z^*)\; e^{-\kappa(N+1/2)|z|^2}$$

THIS IS AN ENTANGLEMENT BREAKING CHANNEL: we can always represent it as a measure and re-prepare channel

Bosonic Gaussian Channels (BGCs)



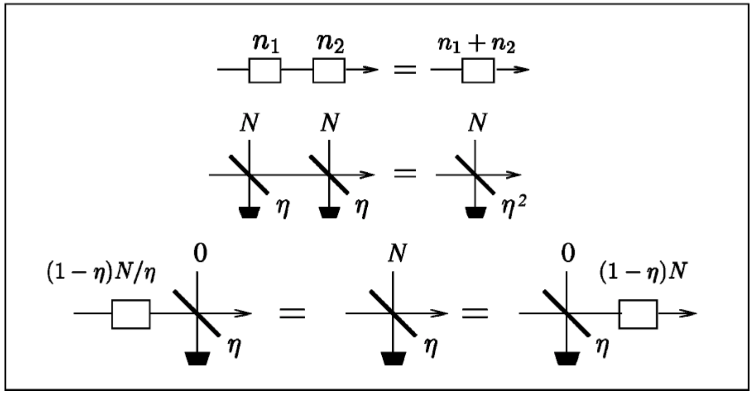
$$\chi'(z) = \chi(z)\, e^{-n|z|^2}$$

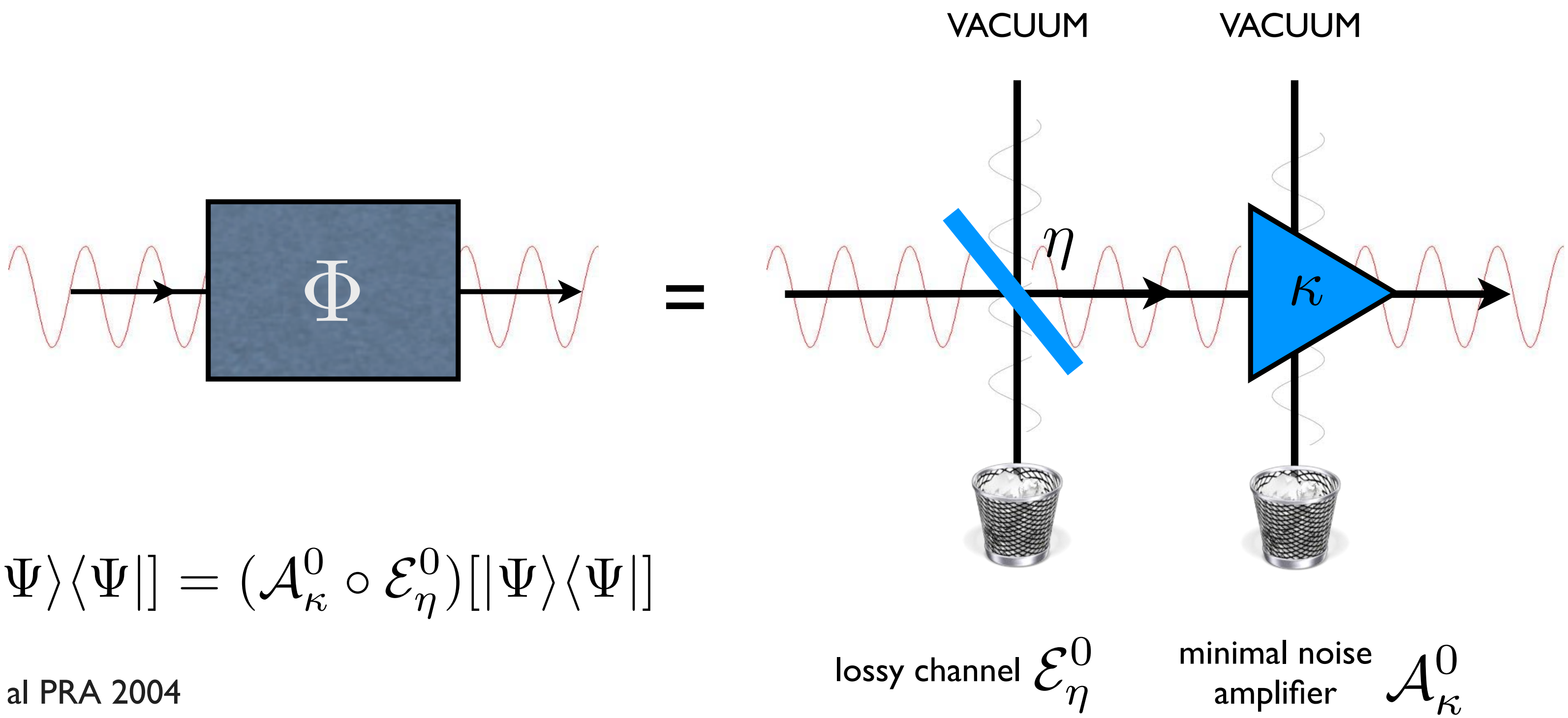


$$\mathcal{N}_n(\rho) = \int d^2\mu\ P_n(\mu)\ D(\mu)\rho D^\dagger(\mu)$$

Bosonic Gaussian Channels (BGCs)

This set of maps is closed under channel concatenation (semigroup structure)





$$\Phi[|\Psi\rangle\langle\Psi|] = (\mathcal{A}_\kappa^0 \circ \mathcal{E}_\eta^0)[|\Psi\rangle\langle\Psi|]$$

VG et al PRA 2004
Garcia-Patron et al. PRL 2012

Bosonic Gaussian Channels (BGCs)

CLASSICAL CAPACITY PROBLEM: how much CLASSICAL information can we transfer over these channels?

Holevo, Schumacher, Westmoreland (HSW) theorem

$$C(\Phi) = \lim_{m\to\infty} \frac{1}{m} C_\chi(\Phi^{\otimes m})$$

regularization over channel uses (Hastings 2008)

$$C_\chi(\Psi) = \sup_{\mathrm{ENS}} \left\{ S(\Psi(\rho_{\mathrm{ENS}})) - \sum_j p_j S(\Psi[\rho_j]) \right\}$$

$$\mathrm{ENS} = \{p_j, \rho_j\}$$

Input energy constraint

$$\mathrm{Tr}[a_j^\dagger a_j \rho] \leq E$$

maximum mean energy per channel use

3. "The Conjectures"

“The Conjectures”

Gaussian Additivity Conjecture

“The output Holevo information is additive (i.e. no regularization over is required)”

Optimal Gaussian ensemble Conjecture

“The maximization of C can be performed over the set of Gaussian ensembles”

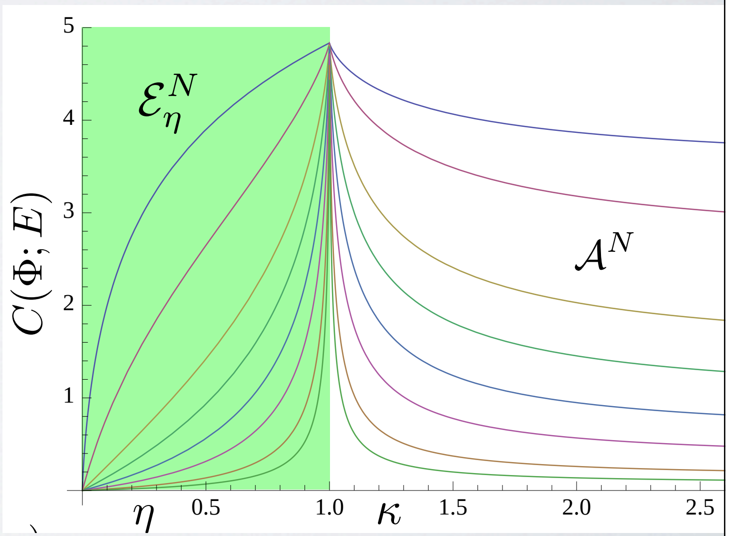
Holevo, Werner PRA 63, 1997

PROVED FOR N=0
(purely lossy channel)

VG et al. PRL 2004

$$C(\mathcal{E}_\eta^N; E) = g(\eta E + (1-\eta)N) - g((1-\eta)N)$$

$$g(x) = (x+1)\log_2(x+1) - x\log_2 x$$



"The Conjectures"

shortcut

shortcut

PROVED FOR N=0
(purely lossy channel)

GIOVANNETTI, GUHA, LLOYD, MACCONE, SHAPIRO, YUEN, PRL 2004

$$C(\mathcal{E}_\eta^0) = \lim_{m\to\infty} \sup_{\mathrm{ENS}} [S([\mathcal{E}_\eta^0]^{\otimes m}(\rho_{\mathrm{ENS}})) - \sum_j p_j S([\mathcal{E}_\eta^0]^{\otimes m}[\rho_j])]/m$$

$$\leqslant \lim_{m\to\infty} [S_{max}([\mathcal{E}_\eta^0]^{\otimes m}) - S_{min}([\mathcal{E}_\eta^0]^{\otimes m})]/m$$

$$\leqslant S_{max}(\mathcal{E}_\eta^0)$$

THERE EXISTS AN ENSEMBLE
THAT SATURATES THE BOUND!!!!

coherent states

$$ENS = \{p(\alpha), |\alpha\rangle\}$$

gaussian ensemble

$$\mathcal{E}_\eta^0(|\alpha\rangle\langle\alpha|) = |\sqrt{\eta}\alpha\rangle\langle\sqrt{\eta}\alpha| \Rightarrow S(\mathcal{E}_\eta^0(|\alpha\rangle\langle\alpha|)) = 0$$

"The Conjectures"

Minimum Output Entropy Conjecture VG et al. 2004

"The Von Neumann Entropy at the output of the channel is minimized by coherent input states (say the vacuum)"

$\Downarrow$

Optimal Gaussian ensemble Conjecture

"The maximization of C can be performed over the set of Gaussian ensembles"

Gaussian Additivity Conjecture

"The output Holevo information is additive (i.e. no regularization over is required)"

Holevo, Werner PRA 63, 1997

$\Longrightarrow$ **CLASSICAL CAPACITY**

shortcut

$$C(\Phi) = \lim_{m\to\infty} \sup_{\text{ENS}} [S(\Phi^{\otimes m}(\rho_{\text{ENS}})) - \sum_j p_j S(\Phi^{\otimes m}[\rho_j])]/m$$

$$\leqslant \lim_{m\to\infty} [S_{max}(\Phi^{\otimes m}) - S_{min}(\Phi^{\otimes m})]/m$$

If MOE is true then the upper bound coincides with the value attainable by Gaussian encoding

good luck ...

QIT-EQIS ERATO Conference (2003)

"The Conjectures"

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Holevo, Werner PRA 63, 1997

$$\Longrightarrow$$

CLASSICAL CAPACITY

“The Conjectures”

Majorization Conjecture

VG et al. PRA 2004

“The output of coherent states majorize all other output states”

Minimum Output Renyi, Weyl Entropy Conjecture

“Coherent states minimize the ouput Renyi Entropies”

Minimum Output Entropy Conjecture

VG et al. PRA 2004

“The Von Neumann Entropy at the output of the channel is minimized by coherent input states (say the vacuum)”

Entanglement of Formation Formula for two-mode Gaussian states

Optimal Gaussian ensemble Conjecture

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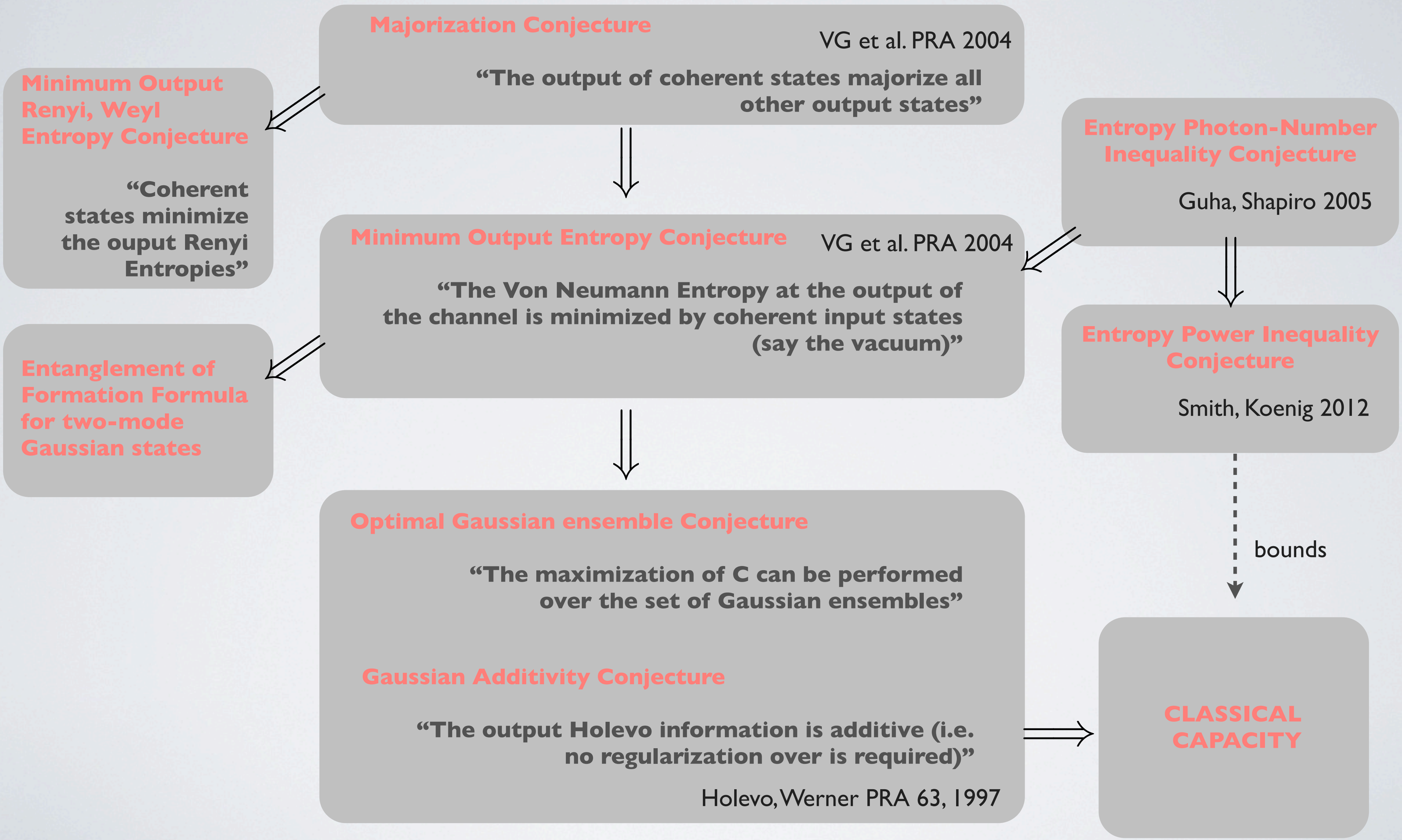
Holevo, Werner PRA 63, 1997

CLASSICAL CAPACITY

"The Conjectures"



"The Conjectures"

december 2013

Majorization Conjecture

VG et al. PRA 2004

"The output of coherent states majorize all other output states"

Minimum Output Renyi, Weyl Entropy Conjecture

"Coherent states minimize the ouput Renyi Entropies"

Entropy Photon-Number Inequality Conjecture

Guha, Shapiro 2005

Minimum Output Entropy Conjecture

VG et al. PRA 2004

"The Von Neumann Entropy at the output of the channel is minimized by coherent input states (say the vacuum)"

Entanglement of Formation Formula for two-mode Gaussian states

Entropy Power Inequality Conjecture

Smith, Koenig 2012

bounds

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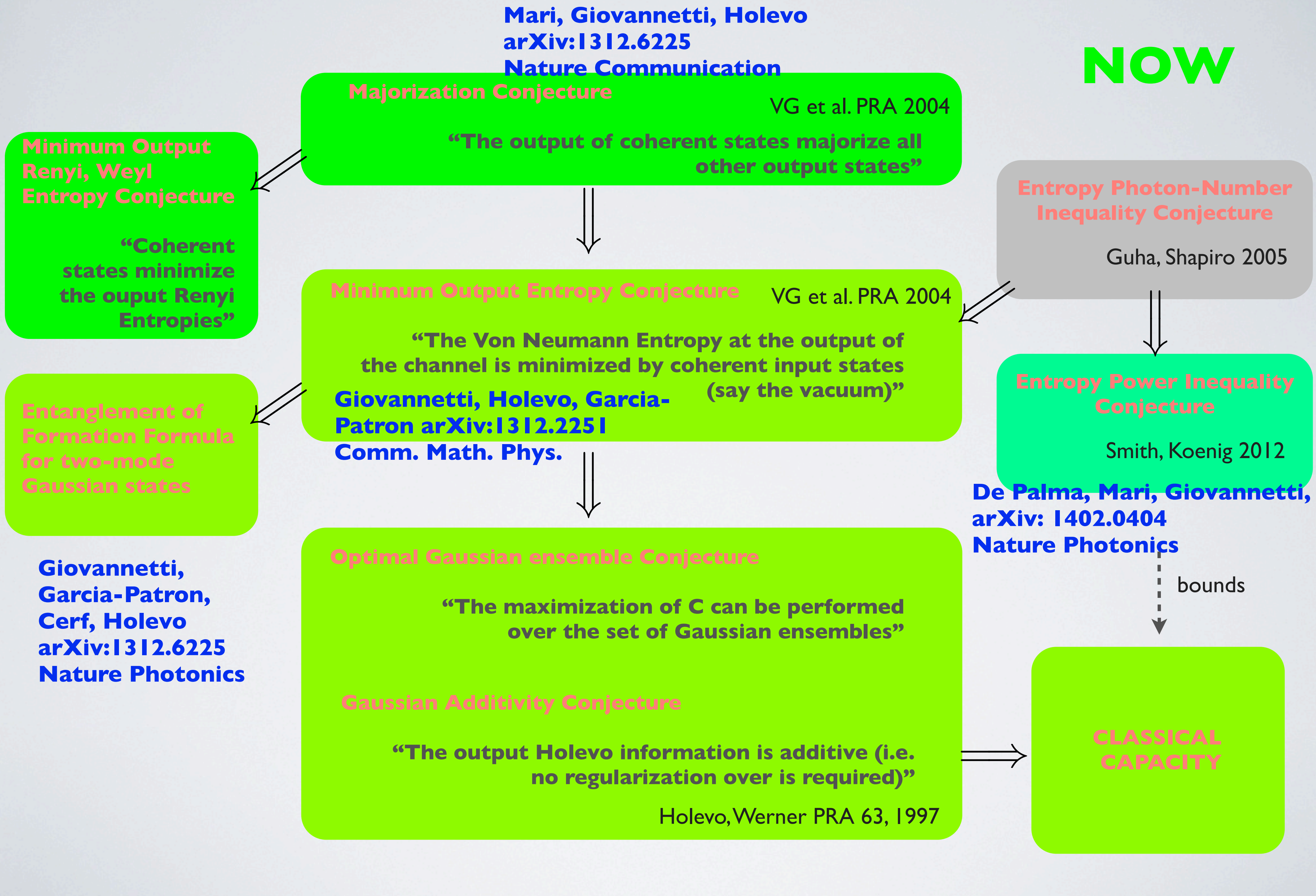
Holevo, Werner PRA 63, 1997

CLASSICAL CAPACITY

"The Conjectures"



"The Conjectures"

Minimum Output Entropy Conjecture

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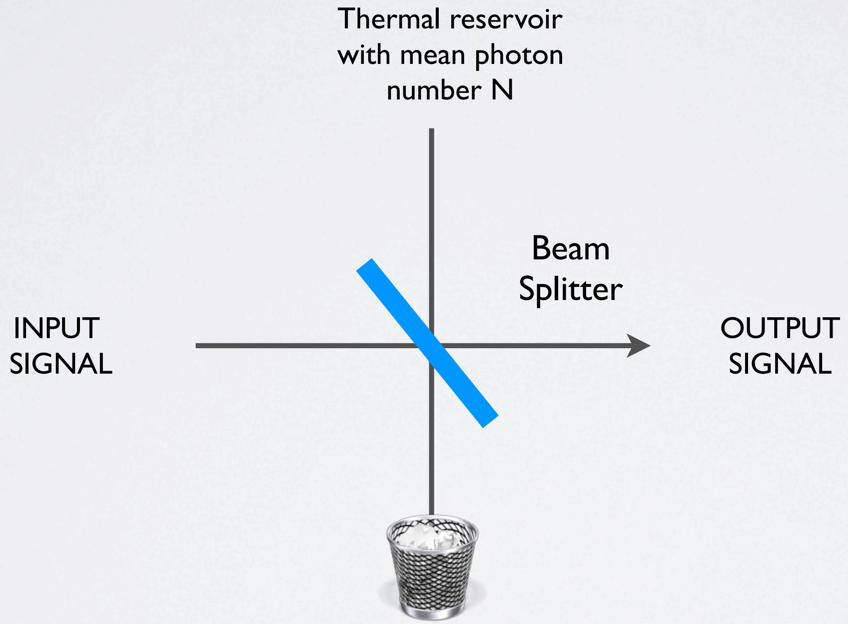
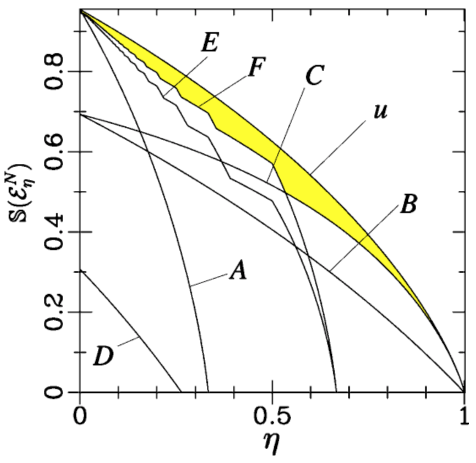
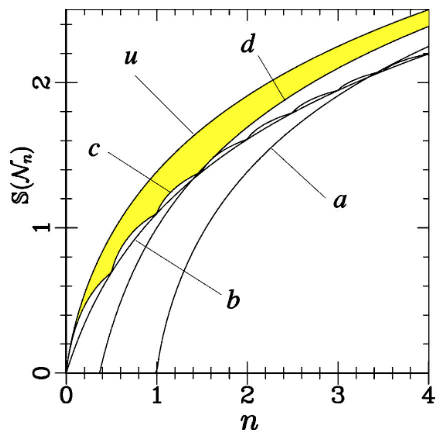


VG, Guha, Lloyd, Maccone, Shapiro, PRA 2004

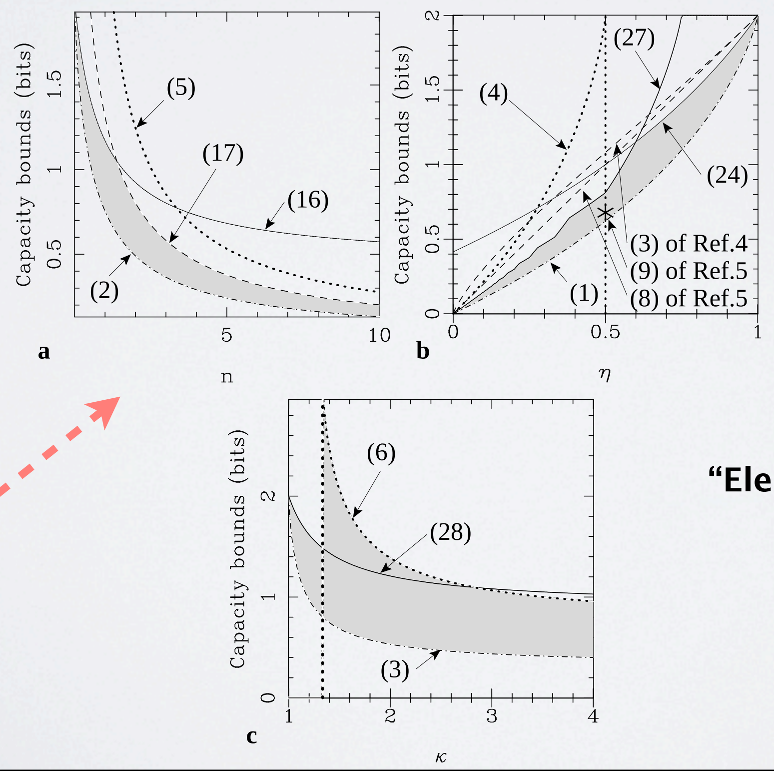
"The Conjectures"





Minimum Output Entropy Conjecture

VG, Guha, Lloyd, Maccone, Shapiro, PRA 2004



VG, Lloyd, Maccone, Shapiro
Nat Phot 2013.

"Electromagnetic channel capacity for practical purposes"

Entropy Power Inequality Conjecture

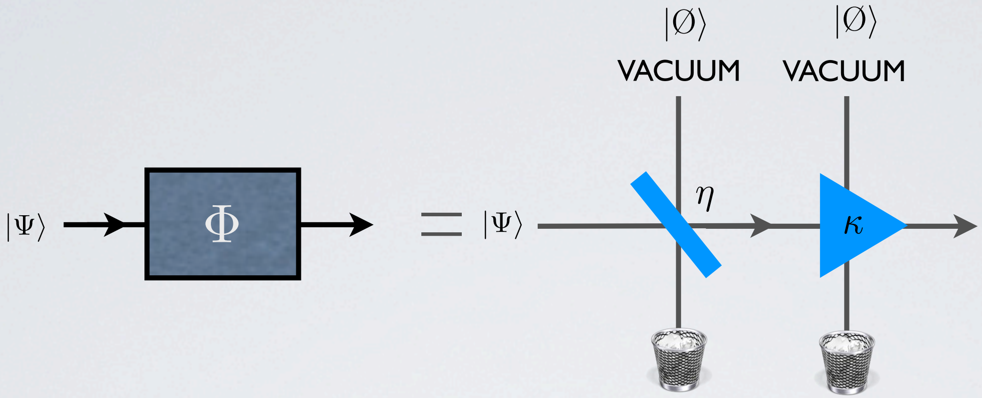
Smith, Koenig 2012

Giovannetti, Holevo, Garcia-Patron arXiv: 1312.2251 COMM. MATH. PHYS.

The
solution
in
5 (simple)
STEPS

Giovannetti, Holevo, Garcia-Patron arXiv: 1312.2251 COMM. MATH. PHYS.

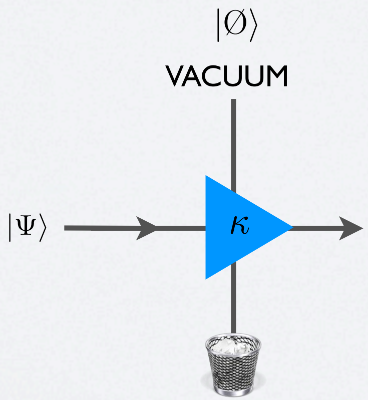
STEP 1



$$\Phi[|\Psi\rangle\langle\Psi|] = (\mathcal{A}_\kappa^0 \circ \mathcal{E}_\eta^0)[|\Psi\rangle\langle\Psi|]$$

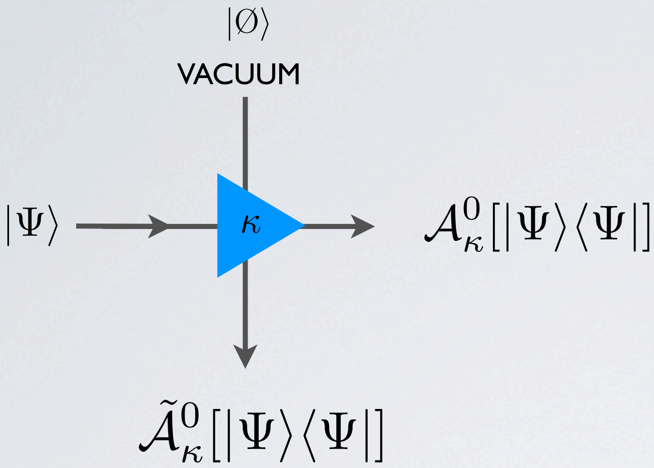
SINCE VACUUM GOES TO THE VACUUM UNDER PURELY LOSSY CHANNEL, PROVING MOE FOR THE AMPLIFIER

$$\mathcal{E}_\eta^0[|\varnothing\rangle\langle\varnothing|] = |\varnothing\rangle\langle\varnothing|$$



Solution

Giovannetti, Holevo, Garcia-Patron arXiv: 1312.2251 COMM. MATH. PHYS.



THIS IS A PROPER STINESPRING REPRESENTATION FOR THE CHANNEL: there for pure inputs we have

STEP II

$$S(\tilde{\mathcal{A}}^0_\kappa[|\Psi\rangle\langle\Psi|]) = S(\mathcal{A}^0_\kappa[|\Psi\rangle\langle\Psi|])$$

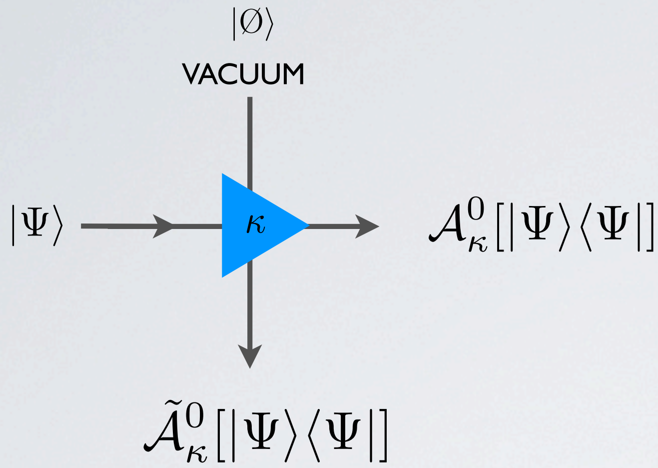
STEP III

... BUT now we can use once more the LOSSY+minimal NOISE AMPLIFIER decomposition to express

$$\tilde{\mathcal{A}}^0_\kappa[|\Psi\rangle\langle\Psi|] = T \circ \mathcal{A}^0_\kappa \circ \mathcal{E}^0_{\eta'}[|\Psi\rangle\langle\Psi|]$$

PHASE CONJUGATION IT DOESN'T CHANGE THE SPECTRUM ... hence the entropy: WE CAN NEGLET IT!

Giovannetti, Holevo, Garcia-Patron arXiv: 1312.2251 COMM. MATH. PHYS.



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PHASE CONJUGATION IT DOESN'T CHANGE THE SPECTRUM ... hence the entropy: WE CAN NEGLET IT!

LUCKY STRIKE

SAME GAIN PARAMETER!!

BINGO!!!!

Giovannetti, Holevo, Garcia-Patron arXiv: 1312.2251 COMM. MATH. PHYS.

Step IV

$$\mathcal{E}^{0}_{\eta'}(|\Psi\rangle\langle\Psi|) = \sum_j p_j |\Psi_j\rangle\langle\Psi_j|$$

$$S(\mathcal{A}^{0}_{\kappa}[|\Psi\rangle\langle\Psi|]) = S(\tilde{\mathcal{A}}^{0}_{\kappa}[|\Psi\rangle\langle\Psi|]) = S(\mathcal{A}^{0}_{\kappa} \circ \mathcal{E}^{0}_{\eta'}[|\Psi\rangle\langle\Psi|])$$

$$= S(\mathcal{A}^{0}_{\kappa}(\sum_j p_j |\Psi_j\rangle\langle\Psi_j|) \geq \sum_j p_j S(\mathcal{A}^{0}_{\kappa}[|\Psi_j\rangle\langle\Psi_j|])$$

$$S(\mathcal{A}^{0}_{\kappa}[|\Psi\rangle\langle\Psi|]) \geq \sum_j p_j S(\mathcal{A}^{0}_{k}[|\Psi_j\rangle\langle\Psi_j|])$$

Giovannetti, Holevo, Garcia-Patron
arXiv:1312.2251

Step V ITERATE THE ARGUMENT q times

$$S(\mathcal{A}^0_\kappa[|\Psi\rangle\langle\Psi|]) \geq \sum_j p_j S(\mathcal{A}^0_\kappa[|\Psi_j\rangle\langle\Psi_j|])$$

$$\sum_j p_j |\Psi_j\rangle\langle\Psi_j| = [\mathcal{E}^0_{\eta'}]^q(|\Psi\rangle\langle\Psi|)$$

$$\lim_{q\to\infty} [\mathcal{E}^0_{\eta'}]^q[|\Psi\rangle\langle\Psi|] = |\emptyset\rangle\langle\emptyset|$$

THE PURELY LOSSY CHANNEL IS MIXING: ITERATING IT MANY TIMES IT BRINGS ALL INPUT STATES TO THE VACUUM ....

$$S(\mathcal{A}^0_\kappa[|\Psi\rangle\langle\Psi|]) \geq S(\mathcal{A}^0_\kappa[|\emptyset\rangle\langle\emptyset|])^*$$

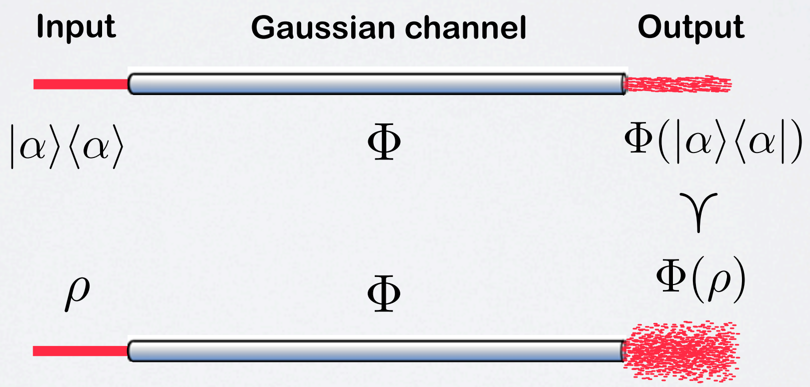
QED

* needs to enforce continuity condition (use the mean energy constraint)

Solution

Mari, Giovannetti, Holevo arXiv:1312.2251
NATURE COMMUNICATION

MAJORIZATION



Giovannetti, Holevo, Mari arXiv: 1405.4066

a consequence: generalization of the Lieb, Solovej inequality

$$\int f(p_\rho(z))\frac{d^{2s}z}{\pi^s} \geq \int f(p_{|\alpha\rangle\langle\alpha|}(z))\frac{d^{2s}z}{\pi^s}$$

$$p_\rho(z) = \mathrm{Tr}[\rho D(z)\rho_0 D^\dagger(z)]$$

$f(x)$ CONCAVE

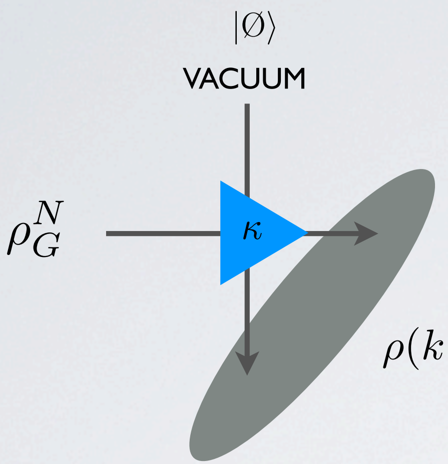
PHASE INVARIANT GAUSSIAN STATE

TAKING $\rho_0 = |\emptyset\rangle\langle\emptyset|$ THIS IS THE HUSIMI DISTRIBUTION

Lieb and Solovej (2012)
Lieb (1978)

Solution

Giovannetti, Garcia-Patron, Cerf, Holevo
arXiv:1312.6225 Nature Photonics



$$\rho(k,N) = U_k(\rho_G^N \otimes |0\rangle\langle 0|)U_k^\dagger$$

$$EoF(\rho(k,N)) \leq EoF(\rho(0,N)) = g(k-1)$$

Giedke, Wolf, Kruger, Werner, Cirac PRL 2003

$$EoF(\rho(k,N)) = \inf_{p_j,|\psi_j\rangle} \sum_j p_j S(\mathcal{A}_k(|\psi_j\rangle\langle\psi_j|)) \geq S(\mathcal{A}_k(|0\rangle\langle 0|)) = EoF(\rho(0,N))$$

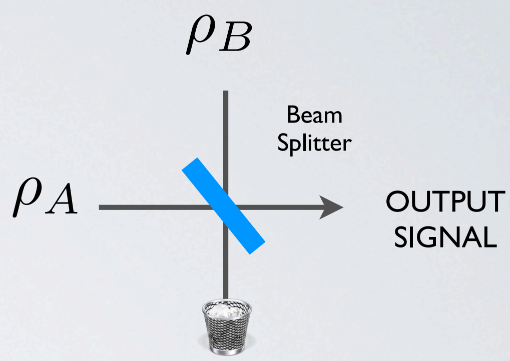
Matsumoto, Shimono, Winter, CMP 2004

Solution

DePalma, Mari, Giovannetti, arXiv: 1402.0404 Nature Photonics

Entropy Power Inequality Conjecture Smith, Koenig 2012

Given SA and SB input entropies of the device the following inequality holds



$$S(\rho_A) = S_A$$

$$S(\rho_B) = S_B$$

$$e^{S_C} \geq \eta e^{S_A} + (1-\eta) e^{S_B}$$

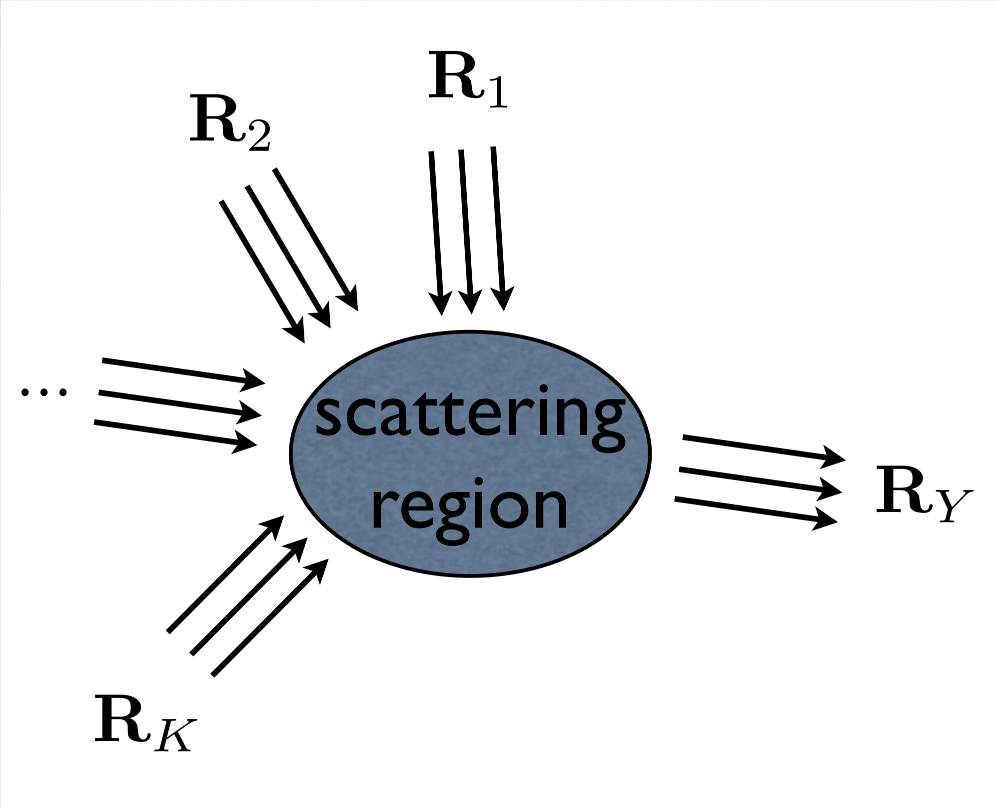
PROVED FOR $\eta = 1/2$ IN Smith, Koenig 2012

PROOF EXTENDED FOR ALL η AND GENERALIZED TO AMPLIFIER CHANNEL TO

$$e^{S_C} \geq \kappa e^{S_A} + (\kappa - 1) e^{S_B}$$

Solution

DePalma, Mari, Lloyd,Giovannetti, arXiv: 1408.0560 (multimode)



$$\exp\left(\frac{1}{n}S_Y\right) \geq \sum_{\alpha=1}^{K} |\det M_\alpha|^{\frac{1}{n}} \exp\left(\frac{1}{n}S_\alpha\right)$$

Conclusions and Perspectives

A BUNCH OF CONJECTURES ON CV SYSTEMS HAVE BEEN RECENTLY SOLVED.

THE STRONGEST OF THEM IS THE MAJORIZATION CONJECTURE, e.g.

- STRONG CONVERSE FOR GAUSSIAN CHANNELS
  BARDHAN, GARCIA-PATRON, WILDE, WINTER arXiv:1401.4161

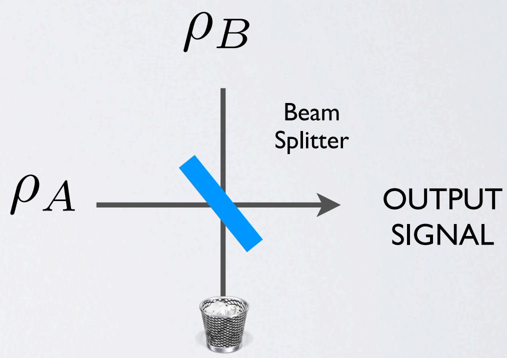
- CLASSICAL CAPACITY OF MEMORY GAUSSIAN BOSONIC CHANNELS
  DEPALMA, MARI, GIOVANNETTI arXiv:1404.1767

OPEN QUESTIONS:

i. STILL ON THE CHASE for the ENTROPY PHOTON NUMBER INEQUALITY CONJECTURE ....

$$N_C \geq \eta N_A + (1-\eta) N_B$$

photon numbers associated with the
Guassian state which have
the SAME entropy of the state



ii. CONTINUITY ....

iii. CAPACITY FORMULAS FOR NON PHASE-INVARIANT CHANNELS

A solution of the Gaussian optimizer conjecture
V. Giovannetti, A. S. Holevo, R. Garcia-Patron
arXiv:1312.2251
to appear in Comm Math Phys

Quantum state majorization at the output of bosonic Gaussian channels
Andrea Mari, Vittorio Giovannetti, Alexander S. Holevo
arXiv:1312.3545
Nature Communication

Majorization and additivity for multimode bosonic Gaussian channels
Vittorio Giovannetti, Alexander S. Holevo, Andrea Mari
arXiv:1405.4066

Ultimate communication capacity of quantum optical channels by solving the Gaussian minimum-entropy conjecture
V. Giovannetti, R. Garcia-Patron, N. J. Cerf, A. S. Holevo
arXiv:1312.6225
to appear in Nature Photonics

Entropy Power Inequality for Bosonic Quantum Systems
Giacomo De Palma, Andrea Mari, Vittorio Giovannetti
arXiv:1402.0404
to appear in Nature Photonics

The multi-mode quantum Entropy Power Inequality
Giacomo De Palma, Andrea Mari, Seth Lloyd, Vittorio Giovannetti
arXiv:1408.0404

THANK YOU!!!

Mari, Giovannetti, Holevo arXiv: 1312.2251

repeat the same procedure for arbitrary (strictly) concave functionals of the output states of the channel ...

STEPS I, II, III, IV as before ....

$$\mathcal{F}(\mathcal{A}^0_\kappa[|\Psi\rangle\langle\Psi|]) = \mathcal{F}(\mathcal{A}^0_\kappa \circ \mathcal{E}^0_{\eta'}[|\Psi\rangle\langle\Psi|]) \geq \sum_j p_j \mathcal{F}(\mathcal{A}^0_\kappa[|\Psi_j\rangle\langle\Psi_j|])$$

IF $|\Psi\rangle$ MINIMIZE THE FUNCTIONAL SO ALSO $\mathcal{E}^0_{\eta'}(|\Psi\rangle\langle\Psi|) = \sum_j p_j |\Psi_j\rangle\langle\Psi_j|$ MUST DO THE SAME.

BUT $\mathcal{F}$ IS STRICTLY CONCAVE, HENCE $\mathcal{E}^0_{\eta'}[|\Psi\rangle\langle\Psi|]$ MUST BE PURE.

THE ONLY STATES WHICH REMAIN PURE UNDER A LOSSY MAP ARE THE COHERENT STATES

Aharanov et al. (1966)
Asboth et al. (2005)
Jiang et al. (2013)

QED