Knots, computation and materials

Jiannis K. Pachos

Introduction
Topo Systems
Jones polynomials
Anyons
Ising & Fibonacci

Isfahan, September 2014
Computers

Antikythera mechanism

Analogue computer

Robotron Z 9001

Digital computer: 0 & 1
Quantum computers: Why?

- **Computational complexity**
  
  Problems that can be solved in:
  
  - polynomial time (easy)
  - exponential time (hard)
  
  as a function of input size.

- **Classical computers:**

  - **P:** polynomially easy to solve
  - **NP:** polynomially easy to verify solution

- **BQP:** polynomially easy to solve with QC
Quantum computers: Why?

• **Factoring**

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\begin{align*}
18070820886874048059516561644059055662781025167694013491701270214 \\
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\times \\
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\end{align*}
\]

Quantum hackers *exponentially* better than classical hackers!

• **Searching** objects: where is $\diamondsuit$?

• **Errors** during QC are too catastrophic.
Topology promises to solve the problem of errors that inhibit the experimental realisation of quantum computers...

...and it is a lot of fun :-(
Geometry - Topology

- **Geometry**
  - Local properties of object
- **Topology**
  - Global properties of object
Topology of knots and links

Are two knots equivalent?

- Algorithms exist from the '60s
- Extremely time consuming...
- Common problem (speech recognition, ...)
- Mathematically Jones polynomials can recognise if two knots are inequivalent.
Topological quantum effects

Aharonov-Bohm effect

**Magnetic flux** $\Phi$ and charge $e$

$$\left| \Psi(x) \right\rangle \rightarrow e^{ine\Phi} \left| \Psi(x) \right\rangle$$

The phase is a function of winding number $N$

**Topological effect:** $n$ is the integer number of rotations
Particle statistics

Exchange two identical particles:

\[ \Psi(x_1, x_2) = \Psi(x_2, x_1) \]

Statistical symmetry:
Physics stays the same, but \(|\Psi\rangle\) could change!

\[ 2 \times \begin{array}{c} \text{\(x_1\)} \quad \text{\(x_2\)} \end{array} = \begin{array}{c} \text{\(x_2\)} \quad \text{\(x_1\)} \end{array} \]
Anyons and statistics

**Bosons**  \[ |\Psi\rangle \rightarrow |\Psi\rangle \]

**Fermions** \[ |\Psi\rangle \rightarrow e^{i2\pi} |\Psi\rangle \]

3D

Anyons: vortices with flux & charge (fractional).

Aharonov-Bohm effect \(\Leftrightarrow\) Berry Phase.

2D

Anyons: vortices with flux & charge (fractional).
Anyonic properties can be found in 2-dimensional topological physical systems:

- Fractional quantum Hall effect
- Topological insulators
- Cold-atom systems

\[ |\Psi\rangle \rightarrow e^{i2\phi} |\Psi\rangle \]
\[ |\Psi\rangle \rightarrow B |\Psi\rangle \]

Anyons

[G. Palumbo & JKP, “C-S from lattice”, PRL 2013]
Anyons, statistics and knots

Initiate: Pair creation of anyons

Measure: do they fuse to the vacuum?

\[ |\Psi_{\text{output}}\rangle = B_n \ldots B_2 B_1 |\Psi_{\text{input}}\rangle \]
Anyons, statistics and knots

\[ |\Psi_{\text{output}}\rangle = B_n ... B_2 B_1 |\Psi_{\text{input}}\rangle \]
Assume I can generate anyons in the laboratory.

- The state of anyons is efficiently described by their world lines.
- Creation, braiding, fusion.
- The final quantum state of anyons is invariant under continuous deformations of strands.
The Reidemeister moves

Theorem:

Two knots can be deformed continuously one into the other iff one knot can be transformed into the other by local moves:

(I)

(II)

(III)
Skein relations

\[ A \cdot A = -2 \]

\[ \frac{1}{A} \cdot \frac{1}{A} = \frac{1}{A^2} \]

\[ -A^2 - \frac{1}{A^2} = d \]
Reidemeister move (II) is satisfied. Similarly (III).
The Jones polynomial algorithm

8.5 Example II: Jones polynomials from Chern-Simons theories

We now consider the link with two components, as it involves a single twist. Similarly for the inverted twisting, some simple links. Our first example, measuring the work qubit in the $L$ direction of the Bloch sphere finally gives the real $M = \mathbb{R}^3$. As it is a single twist of a simple loop it gives $w(L) = w(L') = 1$. Hence they have the same Jones polynomials for any other hand, so $V = A_2 V A_3$.

To evaluate the Jones polynomials of these links we need to apply relation (8.7) that states

$$\langle L \rangle = A \langle L \rangle + A^{-1} \langle \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \rangle = A + dA^{-1} = (-A)^{-3}$$

$$\langle L \rangle = A \langle L \rangle + A^{-1} \langle \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \rangle = Ad + A^{-1} = (-A)^3$$

$$\langle L \rangle = A \langle L \rangle + A^{-1} \langle \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \rangle = -A^4 - A^{-4}$$
The Skein relations give rise to the Kauffman bracket:

$$\text{Skein}(\includegraphics[width=1cm]{knot}) = \langle L \rangle (A)$$

To satisfy move (I) one needs to define the Jones polynomial:

$$V_L(A) = (-A)^{3w(L)} \langle L \rangle (A)$$

$w(L)$ is the writhe of link. For an oriented link it is the sum of the signs for all crossings:

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**Jones polynomial**

The Skein relations give rise to the **Kauffman bracket**:  
\[
\text{Skein}(\includegraphics[width=1cm]{skein_relation}) = \langle L \rangle(A)
\]

To satisfy move (I) one needs to define the **Jones polynomial**:

\[
V_L(A) = (-A)^{3w(L)} \langle L \rangle(A)
\]

\[
\begin{align*}
\includegraphics[width=1cm]{move_1} & = A + \frac{1}{A} \\
\includegraphics[width=1cm]{move_2} & = (dA + \frac{1}{A}) \\
\includegraphics[width=1cm]{move_3} & = -A^3
\end{align*}
\]

\[w = -1\]
If two links have different Jones polynomials then they are inequivalent.

\[ \Rightarrow \text{use it to distinguish links} \]

Jones polynomials keep:

\[ \text{only topological information, no geometrical} \]
Jones polynomial from anyons

Braiding evolutions of anyonic states:

\[ |\Psi_{\text{final}}\rangle = B_2 B_1 |\Psi_{\text{initial}}\rangle \]

\[ \langle \Psi_{\text{initial}} | \Psi_{\text{final}} \rangle = \frac{1}{d^{n/2-1}} \langle L(\vec{B}) \rangle \]

- Simulate the knot with braiding anyons
- Translate it to circuit model:
  \( \leftrightarrow \) find trace of matrices
Evaluating Jones polynomials is a \#P-hard problem.

Belongs to BQP class.

With quantum computers it is polynomially easy to approximate with additive error.

[Freedman, Kitaev, Larsen, Wang (2002); Aharonov, Jones, Landau (2005); Kauffman, Glaser et al. (2009); Kuperberg (2009)]
Summary

Jones polynomials are used for quantum applications:
- encrypt quantum information
- quantum money
- ...

Topological systems that can support anyons are currently engineered...

http://quantum.leeds.ac.uk/~jiannis
Combining physics, mathematics and computer science, topological quantum computation is a rapidly expanding research area focused on the exploration of quantum evolutions that are immune to errors. In this book, the author presents a variety of different topics developed together for the first time, forming an excellent introduction to topological quantum computation.

The workings of anyonic systems, their properties and their computational power are presented in a pedagogical way. Relevant calculations are fully explained, and numerous worked examples and exercises support and aid understanding. Special emphasis is given to the motivation and physical intuition behind every mathematical concept.

Demystifying difficult topics by using accessible language, this book has broad appeal and is ideal for graduate students and researchers from various disciplines who want to get into this new and exciting research field.

Jiannis K. Pachos is a Reader in the School of Physics and Astronomy at the University of Leeds, UK. He works on a variety of research topics, ranging from quantum field theory to quantum optics. Dr Pachos is a University Research Fellow of the Royal Society.
for your great kindness in the matter of the names respecting which I applied to you; but I regret I have not yet had an opportunity of meeting you last Saturday at Kensington and therefore delayed replying to your obligations.

I have taken your advice and the names and the cations and other cathode anode cations and ions. The last I shall have but little occasion for. I had some late objections made to them here and found myself very much in the condition of the man with his boy and wife. He tried to please every body; but when
Inception of Anyonic Models

1. Take a certain number of different anyons $1, a, b, ...$
   the vacuum $(1)$ and one or more non-trivial particles

2. Define fusion rules between them
   $1 \times a = a$, $a \times b = c + d + ...$, $a \times a = 1 + ...$
   The vacuum acts trivially. Each particle has an anti-particle (might be itself or not).

   - Abelian anyons $a \times b = c$
   - Non-Abelian anyons $a \times b = c + d + ...$
Braiding and Fusion properties

• The action of braiding of two anyons depends on their fusion outcome:

\[ R^c_{ab} \] is a phase factor

• Changing the order of fusion is non-trivial:

\[ \sum_j = R^c_{ab} \]

\[ = \sum_j (F^d_{abc})_{ij} \]
The braid group $B_n$

The braid group $B_n$ has elements $b_1, b_2, \ldots, b_{n-1}$ that satisfy:

\[ b_i b_j = b_j b_i, \quad \text{for } |i - j| \geq 2 \]

\[ b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \quad \text{for } 1 \leq i < n \]

Pictorially:

\[ b_i b_j = b_j b_i \]

\[ b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \]
3. The F and B matrices are determined from the **Pentagon** and **Hexagon identities**

\[
(F_{5a34}^b)^c_{(a)} = \sum_e (F_{234}^c_d e) (F_{1e4}^b_d) (F_{123}^a)_{(a)}
\]
3. The F and B matrices are determined from the Pentagon and Hexagon identities

$$\sum_b \left( F_{231}^4 \right)_b R_{1b}^4 \left( F_{123}^4 \right)_a = R_{13}^c \left( F_{213}^4 \right)_a R_{12}^a$$
Ising Anyons

Consider the particles: 1, σ and ψ

Fusion rules: \( \sigma \times \sigma = 1 + \psi \), \( \psi \times \psi = 1 \), \( \sigma \times \psi = \sigma \)

\[
\begin{array}{cccccccc}
\sigma & \sigma & \sigma & \sigma & \sigma & \sigma & \sigma & \sigma \\
\sigma & 1 & \sigma & 1 & \sigma & \cdots & 1 \\
\psi & \sigma & \psi & \sigma & \cdots & \cdots \\
1 & \sigma & \cdots & \cdots \\
\psi & \sigma & \cdots & \cdots \\
\end{array}
\]

\[d_n = 2^{n/2}\] increase in dim of Hilbert space
Ising Anyons

Consider the particles: 1, $\sigma$ and $\psi$

Fusion rules: $\sigma \times \sigma = 1 + \psi$, $\psi \times \psi = 1$, $\sigma \times \psi = \sigma$

All these states span the fusion Hilbert space.

$|\psi\rangle = |1,1,\ldots\rangle$

$|\psi\rangle = |1,\psi,\ldots\rangle$

Braiding neighboring anyons transforms states
Ising Anyons

Consider the particles: 1, $\sigma$ and $\psi$

Fusion rules: $\sigma \times \sigma = 1 + \psi$, $\psi \times \psi = 1$, $\sigma \times \psi = \sigma$

From 5-gon and 6-gon identities we have:

$$F_{\sigma \sigma \sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H$$

Rotation of basis states
Ising Anyons

**Braiding**

\[ R^{1}_{\sigma\sigma} = e^{-i\pi/8} \] and \[ R^{\psi}_{\sigma\sigma} = ie^{-i\pi/8} \]

\[ \Rightarrow R_{\sigma\sigma} = e^{-i\pi/8} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \]

\[ (R_{\sigma_1\sigma_2})^2 \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{array} \begin{array}{c} 1 \\ \sigma_4 \end{array} = (R^{1}_{\sigma_1\sigma_2})^2 \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{array} \begin{array}{c} 1 \\ \sigma_4 \end{array} + (R^{\psi}_{\sigma_1\sigma_2})^2 \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{array} \begin{array}{c} \psi \\ \sigma_4 \end{array} \]

\[ = e^{-i\pi/4} \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{array} \begin{array}{c} 1 \\ \sigma_4 \end{array} - e^{-i\pi/4} \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{array} \begin{array}{c} \psi \\ \sigma_4 \end{array} \]

\[ = e^{-i\pi/4} \]

\[ H\sigma^z H = \sigma^x \]

Clifford group: non-universal!
Ising Anyons

Qubit initialization:

\[ \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} & & & \sigma & & \sigma & & \sigma & \end{array} \]

\[ \begin{array}{c|c|c|c|c|c|c|c|c|c|c} \sigma & & & & & & & & & & & \\
\psi & & & & & & & & & & & \\
\end{array} \]

State \(|0\rangle\) \quad \text{State} \ (|1\rangle)

Measurement: Outcome of pairwise fusion, 1 or \(\psi\)

\[ H\sigma^z H = \sigma^x \]

Gates: Clifford group. Non-universal!
One needs a phase gate: employ interactions between anyons.

Can be employed as a quantum memory.
Assume we can:
- **Create** identifiable anyons
  - vacuum pair creation

- **Braid** anyons
  - Statistical evolution:
    - braid representation $B$

- **Fuse** anyons
  - \( \sigma \times \sigma = 1 + \psi \)

**Fusion Hilbert space:**
\[
|\sigma, \sigma \rightarrow 1\rangle, |\sigma, \sigma \rightarrow \psi\rangle
\]
Fibonacci Anyons

Consider anyons with labels 1 or $\tau$ with the fusion properties: $1 \times 1 = 1$, $1 \times \tau = \tau$, $\tau \times \tau = 1 + \tau$

$\tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau$

$1 1 1 1$

$\tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau$

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$T$ $1$ $T$ $1$ $T$

$T$ $1$ $T$ $1$

$T$ $1$

$d_1 = 1$

$d_2 = 2$

$d_3 = 3$

$d_4 = 5$ Fibonacci sequence!

$d_5 = 8$

$T$

$1$

$T$

Dimension of Hilbert space

$\dim_n \propto \phi^n$

$\phi = \left(1 + \sqrt{5}\right) / 2$

Golden mean
Fibonacci Anyons and QC

Qubit encoding:

State $|0\rangle$

$|T,T\rightarrow 1\rangle = |0\rangle$

$|T,T\rightarrow T\rangle = |1\rangle$

State $|1\rangle$

Evolving a qubit:

$= R_{ab}^c$

$= \sum_j F_{ij}$

Unitaries $B$ and $F$ are dense in $SU(2)$.

[Freedman, Larsen, Wang, CMP 228, 177 (2002)]
Fibonacci Anyons and QC

Qubit encoding:

Evolving a qubit:

\[ = R_{ab}^c \]

\[ = \sum_j F_{ij} \]

Unitaries B and F are dense in SU(2). Extends to SU(d_n) when n anyons are employed.
Fibonacci Anyons and QC

Qubit encoding:

Unitaries B and F are dense in SU(2). Extends to SU($d_n$) when $n$ anyons are employed.
Conclusions

• Topological Quantum Computation promises to overcome the problem of decoherence and errors in the most direct way.

• There is lots of work to be done to make anyons work for us.

• Is it worth it?

Aesthetics says YES!