Why should anyone care about computing with anyons?

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Toric Code
Non-Abelian
Topological Entropy
Errors
Outlook

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Anyons and Quantum Computation

• Error correction needs a huge overhead.

• Instead of performing active error correction let physics do the job.

• Perform QC in a physical medium that is gapped and highly correlated:
  • Energy penalty for errors (gapped).
  • Make logical errors non-local (very unlikely).

• Similar to quantum error correction, but without active control.
Consider the lattice Hamiltonian

\[ H = -\sum_{p} Z_{p_1}Z_{p_2}Z_{p_3}Z_{p_4} - \sum_{v} X_{v_1}X_{v_2}X_{v_3}X_{v_4} \]

Spins on the edges.

Two different types of interactions: $ZZZZ$ or $XXXX$ acting on plaquettes and vertices respectively.

The four spin interactions involve spins of the same vertex/plaquette.
Consider the lattice Hamiltonian

\[ H = - \sum_p Z_{p1} Z_{p2} Z_{p3} Z_{p4} - \sum_v X_{v1} X_{v2} X_{v3} X_{v4} \]

**Good quantum numbers:**

\[ [H, Z_{p1} Z_{p2} Z_{p3} Z_{p4}] = 0 \]
\[ [H, X_{v1} X_{v2} X_{v3} X_{v4}] = 0 \]
\[ (X_{v1} X_{v2} X_{v3} X_{v4})^2 = 1 \]
\[ (Z_{p1} Z_{p2} Z_{p3} Z_{p4})^2 = 1 \]

\[ \Rightarrow \text{eigenvalues of XXXX and ZZZZ: } \pm 1 \]

Also Hamiltonian exactly solvable:

\[ [X_{v1} X_{v2} X_{v3} X_{v4}, Z_{p1} Z_{p2} Z_{p3} Z_{p4}] = 0 \]
Consider the lattice Hamiltonian

\[ H = - \sum_p Z_{p_1} Z_{p_2} Z_{p_3} Z_{p_4} - \sum_v X_{v_1} X_{v_2} X_{v_3} X_{v_4} \]

Indeed, the ground state is:

\[ |\xi\rangle = \prod_v \frac{1}{\sqrt{2}} (1 + X_{v_1} X_{v_2} X_{v_3} X_{v_4}) |00...0\rangle \]

The $|00...0\rangle$ state is a ground state of the $ZZZZ$ term. The $(I+XXXX)$ term projects that state to the ground state of the $XXXX$ term.
Consider the lattice Hamiltonian

\[ H = - \sum_{p} Z_{p1} Z_{p2} Z_{p3} Z_{p4} - \sum_{v} X_{v1} X_{v2} X_{v3} X_{v4} \]

Indeed, the ground state is:

\[
|\xi\rangle = \prod_{v} \frac{1}{\sqrt{2}} (I + X_{v1} X_{v2} X_{v3} X_{v4}) |00...0\rangle
\]

The ground state is a superposition of all X loops.
It is stabilized by the application of all X loop operators.
Equivalently for Z loops.
Toric Code: ECC

- **Excitations** are produced by Z or X rotations of one spin.
- These rotations *anticommute* with the X- or Z-part of the Hamiltonian, respectively.
- Z excitations on $v$ vertices.
- X excitations on $p$ plaquettes.

$X$ and $Z$ excitations behave as anyons with respect to each other.
One can demonstrate the anyonic statistics between X and Z. First create excitations with Z and X rotations.
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\[ |Final\rangle = Z_4 Z_3 Z_2 Z_1 |X\rangle = (Z_4 Z_3 Z_2 Z_1) X_3 |\xi\rangle = -X_3 (Z_4 Z_3 Z_2 Z_1) |\xi\rangle = -|Initial\rangle \]
Anyonic statistics

After a complete rotation of an X anyon around a Z anyon (two successive exchanges) the resulting state gets a phase $\pi$ (a minus sign): hence ANYONS with statistical angle $\pi / 2$

A property we used is that $X_4 X_3 X_2 X_1 \xi = \xi$
One can demonstrate the anyonic statistics between X and Z. First create excitations with Z and X rotations. Then rotate Z excitation around the X one. This results in plaquette operator detecting the X excitation. Gives -1

\[
|\text{Final}\rangle = X_4 X_3 X_2 X_1 |Z\rangle = (X_4 X_3 X_2 X_1) Z_3 |\xi\rangle \\
- Z_3 (X_4 X_3 X_2 X_1) |\xi\rangle = - |\text{Initial}\rangle
\]
Toric Code: Anyons

Hence Toric Code has particles:

1, e (Z), m (X), ε (fermion)

Fusion rules:

\[ e \times e = 1, \ m \times m = 1, \ \varepsilon \times \varepsilon = 1 \]
\[ e \times m = \varepsilon, \ e \times \varepsilon = m, \ m \times \varepsilon = e \]

Fusion moves: F are trivial

Braiding moves R: \[ R_{em}^{\varepsilon} = i, \ R_{\varepsilon\varepsilon}^{1} = -1 \]
Toric Code: Encoding

Toric code as a quantum error correcting code.

Consider periodic boundary conditions: TORUS of size $L$

Errors: Anyons

Error correction: detect anyons/errors and connect shortest distance between the same type of anyons.

$$\prod_v X_{v_1} X_{v_2} X_{v_3} X_{v_4} = \prod_p Z_{p_1} Z_{p_2} Z_{p_3} Z_{p_4} = 1$$
Toric Code: Encoding

Toric code as a quantum error correcting code.

Consider periodic boundary conditions: TORUS of size L

Errors: Anyons

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Logical Gates: non-trivial loops
Toric Code: Encoding

Logical Space and Gates

<table>
<thead>
<tr>
<th>$\Psi_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Psi_2 = C_X^1</td>
</tr>
<tr>
<td>$\Psi_3 = C_X^2</td>
</tr>
<tr>
<td>$\Psi_4 = C_X^2 C_X^1</td>
</tr>
</tbody>
</table>

Can store two qubits and perform Clifford group operations!

Higher genus, $g$, stores $2g$ qubits.
Quantum Double Models

Toric Code is an example of quantum double models. Corresponding group $\mathbb{Z}_2 = \{1, e; e^2 = 1\}$ that gives rise to qubit states $|1\rangle, |e\rangle$.

Imagine a general finite group $G = \{g_1, g_2, ..., g_d\}$ and the corresponding qudit with states $|g_i\rangle$, $i=1,...,d$.

Consider a qudit positioned at each edge of a square lattice.
Define orientation on the lattice:
Upwards and Rightwards
Quantum Double Models

Define operators:

\[ L^g_+ |z\rangle = |gz\rangle, \quad L^g_- |z\rangle = |zg^{-1}\rangle, \quad T^h_+ |z\rangle = \delta_{h,z} |z\rangle, \quad T^h_- |z\rangle = \delta_{h^{-1},z} |z\rangle \]

\[ A(v) = \frac{1}{|G|} \sum_{g \in G} L^g_+ (e_1)L^g_+ (e_2)L^g_- (e_3)L^g_- (e_4), \quad B(p) = \sum_{h_1 \ldots h_4 = 1} T^h_1 (e_1)T^h_2 (e_2)T^h_3 (e_3)T^h_4 (e_4) \]

Hamiltonian and ground state:

\[ H = - \sum_v A(v) - \sum_p B(p) \]

\[ A(v) |\xi\rangle = |\xi\rangle \]

\[ B(p) |\xi\rangle = |\xi\rangle \]
Quantum Double Models

This is also an error correcting code defined from the stabilizer formalism.

The errors are anyons, Abelian or non-Abelian, with the corresponding fusion rules, B and F matrices.

These properties can be explicitly determined.

Examples: \( D(Z_2) \), \( D(Z_2 \times Z_2) \), \( D(S_3) \)

\( S_3 = \{1, x, y, y^2, xy, xy^2; x^2 = 1, y^3 = 1\} \)
Quantum Double Models

This is also an error correcting code defined from the stabilizer formalism.

The errors are anyons, Abelian or non-Abelian, with the corresponding fusion rules, B and F matrices.

Information can be encoded in the fusion space of non-Abelian anyons and manipulated by braiding them.

Realizations:
Josephson junctions, photons, optical lattices,...
From Abelian to Nonabelion

- The scheme: $D(Z_2 \times Z_2)$ [or $D(S_3)$] similar to two toric codes

Hamiltonian bird’s eye view:

- Empty
- Lower only
- Upper only
- Upper and Lower

Vacuum Abelian Anyon

-1

$\mu$

“Non-Abelian” Anyon

$X$
From Abelian to Nonabelian

- Encode information in fusion channels:
  \[ \mu \times \mu = 1, \quad \chi \times \chi = 1 + \mu \]
- Qubit needs four anyons
- Logical \(|0\rangle\) when each pair fuses to the vacuum 1
- Logical \(|1\rangle\) when each pair fuses to \(\mu\)
- 1, \(\mu\) indistinguishable to local operations when dressed with \(\chi\)

- Measurement by fusion
From Abelian to Nonabelian

- Fault-tolerance
  - Phase flips
  - Bit flips
  by non-local operators only
  \[ \text{topo. protection} \]

Energy gap present even during gate operations

- \[ \Delta E \]

- Redundancy and non-locality protects against virtual transitions

- Braiding is only Abelian.
Summary

• Quantum Double models:
  - Toric Code
  - Abelian encoding and quantum computation
  - Non-Abelian models

• Degenerate encoding states

• Energy gap above encoding space

• Manipulations of code space: higher genus surface or with anyons or punctures: encoding Hilbert space becomes larger.
Further

- Detecting topological order
  - Topological entropy
- Errors and topological order
  - Topological memories
- Protection against errors
Topological Entropy

- Pure system $|\xi\rangle$
- Partition in $R$ and $\bar{R}$ with boundary $\partial R$
- Reduced density matrix of $R$:
  $$\rho_R = \text{tr}_\bar{R} |\xi\rangle\langle\xi|$$
- Von Neumann entropy:
  $$S_R = \text{tr}(\rho_R \ln \rho_R)$$
- We expect:
  $$S_R = \alpha |\partial R| + \gamma + \varepsilon (|\partial R|^{-1})$$
- Topological entropy:
  $$\gamma = \ln D, \quad D = \sqrt{\sum_q d_q^2}$$
Topological Entropy

• Consider partition of single system \( \Sigma \):

\[ \Sigma \]

System is \textbf{gapped} \rightarrow finite correlation length
Size of areas \rightarrow \textbf{infinity}

\[ \varepsilon (| \partial R |^{-1}) \rightarrow 0 \]

• Topological entropy [Kitaev & Preskill]:

\[ \gamma = S_A + S_B + S_C - S_{AB} - S_{AC} - S_{BC} + S_{ABC} \]

The area terms disappear! \( S_R = \alpha | \partial R | + \gamma \)
Topological Entropy

- Consider four different partitions:

\[
\begin{align*}
S_A & \quad \Sigma A \\
S_B & \quad \Sigma B \\
S_C & \quad \Sigma C \\
S_D & \quad \Sigma D
\end{align*}
\]

- Topological entropy:

\[
\gamma = -\frac{1}{2} \left[ (S_A - S_B) - (S_C - S_D) \right]
\]

- Only loop contributions survive! [Levin & Wen]
Topological Errors

Errors can appear in the form of virtual anyons:
Topological Errors

Errors can appear in the form of virtual anyons.

They can be avoided by keeping data anyons far apart:

$$P_{\text{error}} \sim e^{-\Delta L/v}$$

Δ: Energy Gap for σ pair creation
L: distance between σ anyons
v: characteristic velocity of anyons
Resilience to Errors

- **Abandon** the idea of separate **subsystems** for qubits. Encode info in **macroscopic degree of freedom** (non-locally).
  Direct observation of anyons does not reveal their total state.
  => **local decoherence** (environment “measures”) does not destroy information.
- **The unitary transformations** resulting from braiding are virtually **errorless**.
Resilience to Errors

- Hamiltonian (energy gap) protects against local perturbations.
- Error correction protects against environmentally induced errors.

Topologically inspired quantum error correction. 0.75% tolerance [Raussendorf & Harrington]
Topological Memory

Can you create a 2D system that resists errors due to temperature for long times?

1) Toric code coupled to bosonic field:
   errors (anyons) attract and annihilate!
   [Hamma, Castelnovo & Chamon]

2) Induce a repulsion between anyons:
   it generates a stable anyonic phase.
   [Chesi, Roethlisberger & Loss]

3) Entropic energy barriers
   [Brown, Al-Shimary, JKP]
Outlook

- **Quantum information** has a lot to offer to the study of topological systems.
- **Topological quantum computation** is a very promising way of storing and manipulating quantum information.
- Research on topological quantum computation has **applications** to many relevant fields of condensed matter, statistical physics, biology,...
- Topological states of matter **NEED** mathematics to be understood.