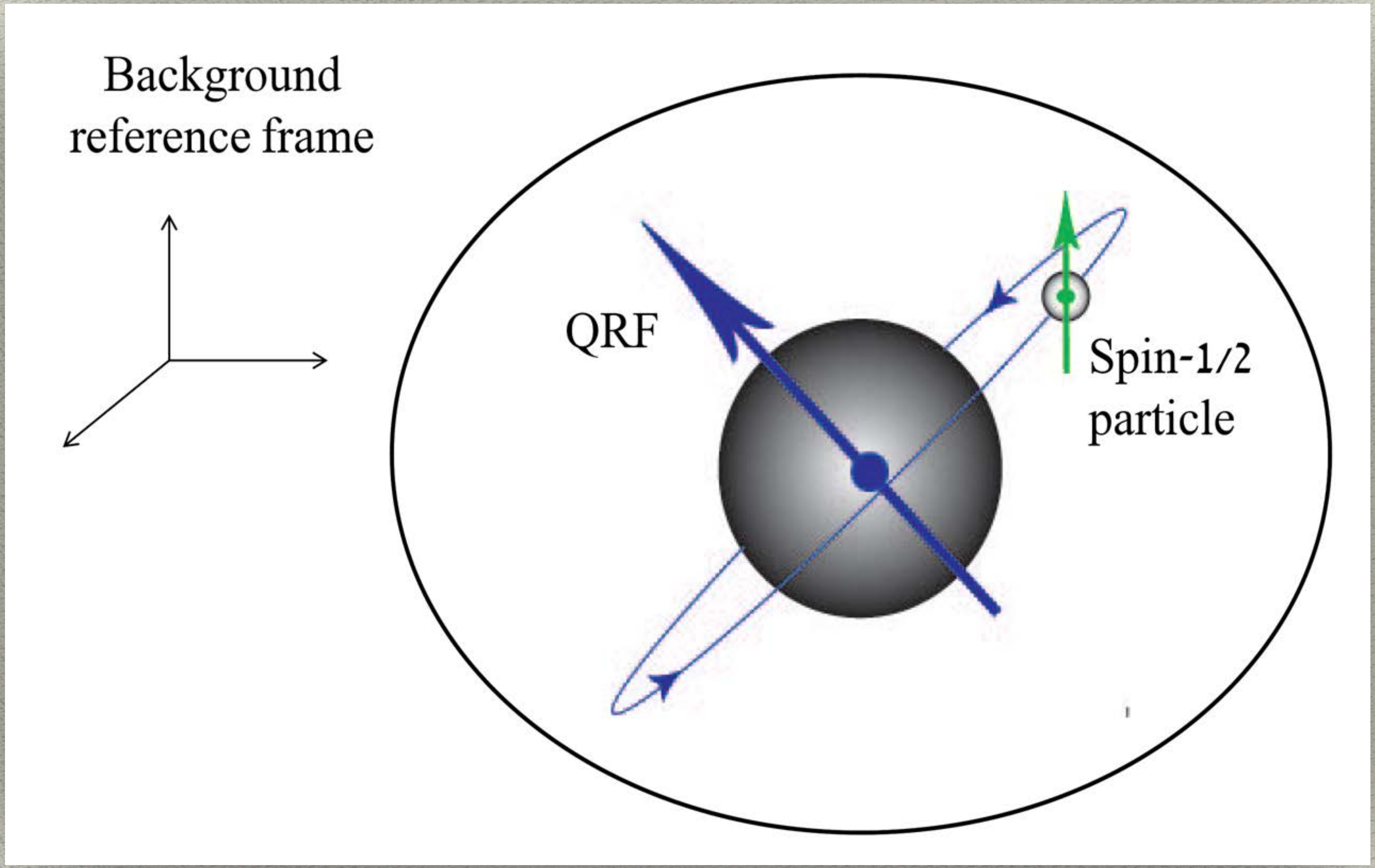
General framework for alignment-free communication



Mehdi Ahmadi

Dominik Šafránek, M. A., and Ivette Fuentes, arxiv:1404.6421 (2014).

Reference frames in quantum theory

- Lacking a shared reference frame is equal to a superselection rule (SSR).
- The lack of a shared reference frame can be treated within the quantum formalism as a form of noise.

[S.D. Bartlett, T. Rudolph and R.W. Spekkens, Rev. Mod. Phys. 79, 555609 (2007)]

- Conditional probability interpretation of time in QM

[D. Page, W.K. Wootters, Phys. Rev. D 27, 2885 (1983).]

(Mis-)aligned reference frames

- Perfect quantum data hiding

[B.M. Terhal, D.P. DiVincenzo, and D.W. Leung, Phys. Rev. Lett. 86, 5807 (2001).]

[F. Verstraete, and J. I. Cirac, Phys. Rev. Lett. 91, 010404 (2003).]

- Arbitrarily secure ancilla-free bit commitment in the presence of SSR

[D.P. DiVincenzo, J.A. Smolin, and B. M. Terhal, New J. Phys. 6, 80 (2004).]

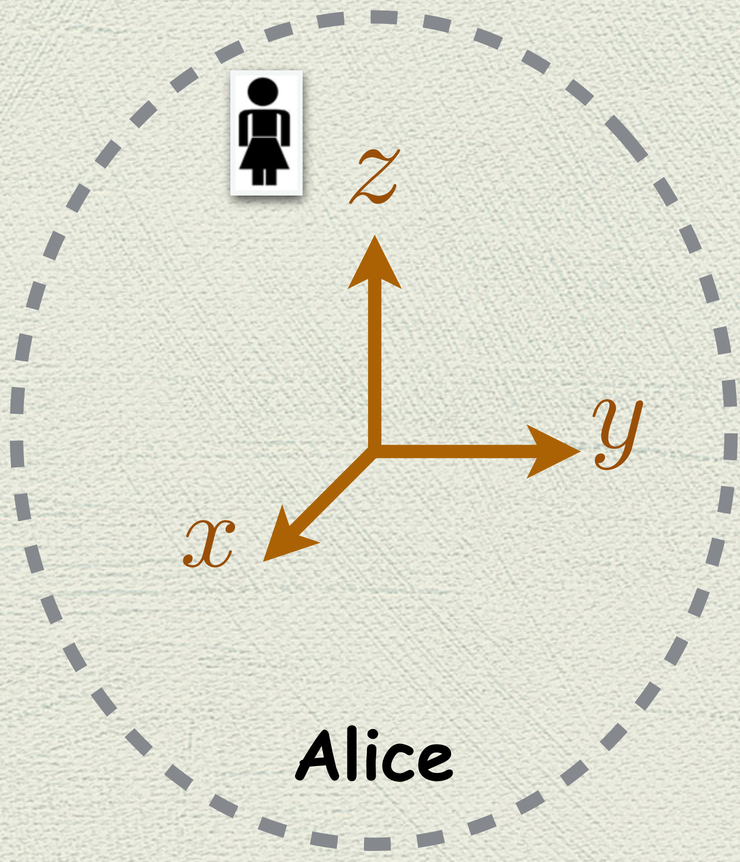
- Alignment-free communication

[S.D. Bartlett, T. Rudolph, R. Spekkens and P. Turner, New J. Phys. 11 063013 (2009).]

Resource theory of asymmetry vs. entanglement

Resource theory	Limitation	Free states	Useful states (resources)
Entanglement	LOCC	Product states	Entangled states
Asymmetry	Symmetric operations	Symmetric states	Asymmetric states

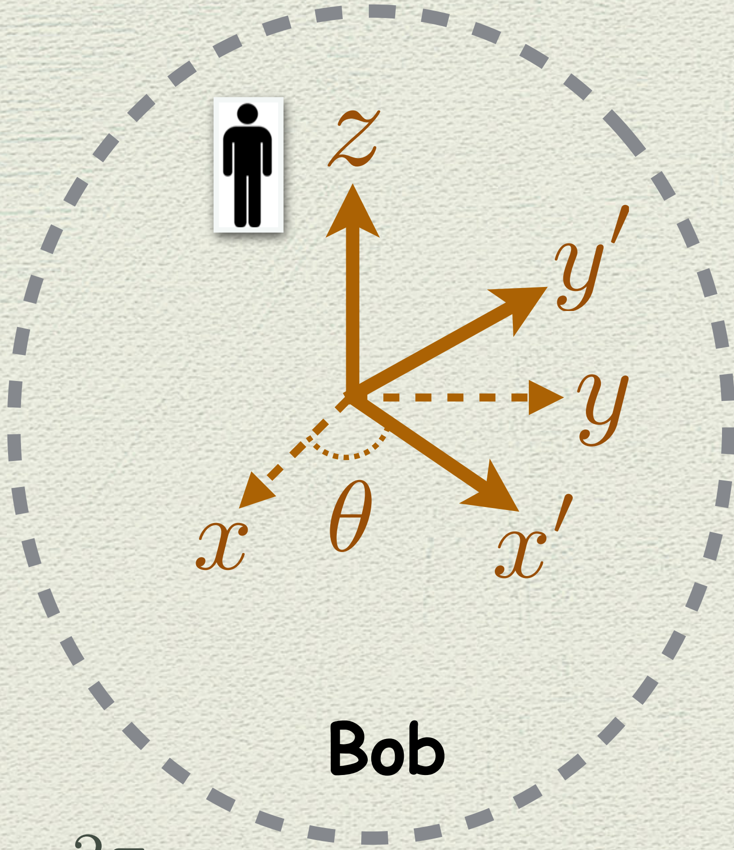
Communication in the absence of aligned reference frames



θ is known

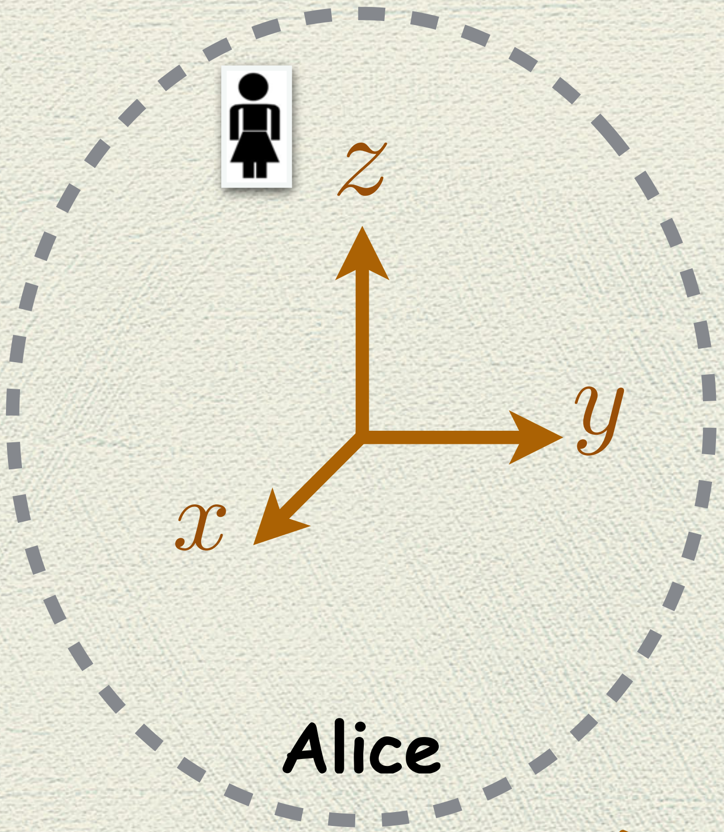
$$\rho_A \longrightarrow U(\theta)\rho_A U^\dagger(\theta)$$

θ is not known

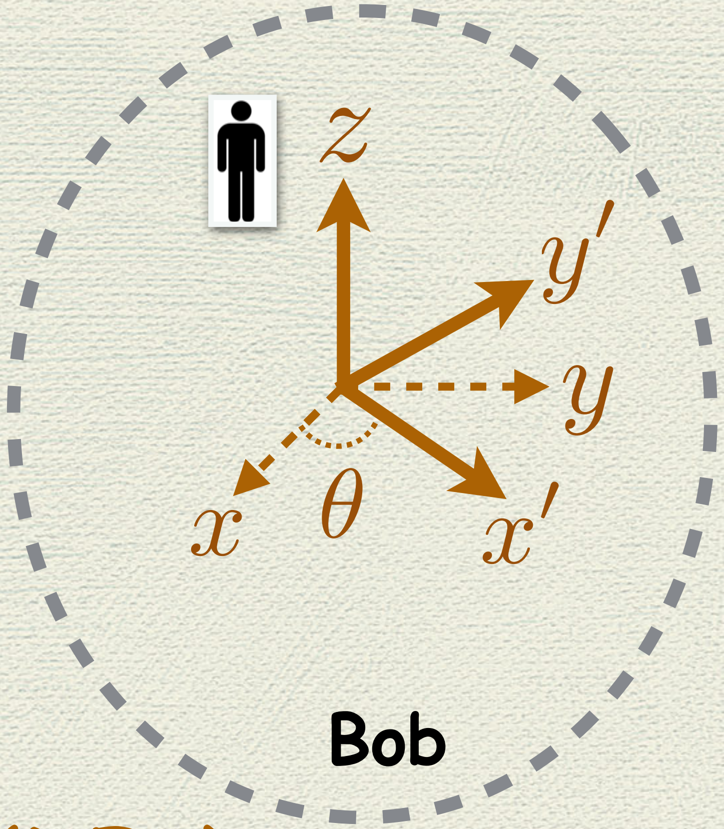


$$\rho_B = \mathcal{G}[\rho_A] = \frac{1}{2\pi}\int_0^{2\pi} U(\theta)\rho_A U^\dagger(\theta)$$

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle) \longrightarrow \mathcal{G}[|\pm\rangle\langle\pm|] = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$



$$|\Psi_\pm\rangle = |\pm\rangle|\Psi\rangle$$



Decoherence-free subspace (DFS)

$|\Psi\rangle = |+\rangle$

$$\longrightarrow \mathcal{G}[|\Psi_\pm\rangle\langle\Psi_\pm|] = \frac{1}{4}(|00\rangle\langle 00| + (|01\rangle \pm |10\rangle)(\langle 01| \pm \langle 10|) + |11\rangle\langle 11|)$$

Bob's set of POVM for Unambiguous discrimination:

$$\{P_+ = \frac{1}{2}(|01\rangle + |10\rangle)(\langle 01| + \langle 01|), P_- = \frac{1}{2}(|01\rangle - |10\rangle)(\langle 01| - \langle 01|),$$

$$P_? = \mathbf{I} - P_+ - P_-\}$$

$$\boxed{p_? = \mathrm{Tr}[P_?\mathcal{G}[|\pm\rangle\langle\pm|]] = \frac{1}{2}}$$

Part Two:
Noisy Quantum Metrology and alignment-free quantum communication

Cramér-Rao bound(CRB)

Classical CRB

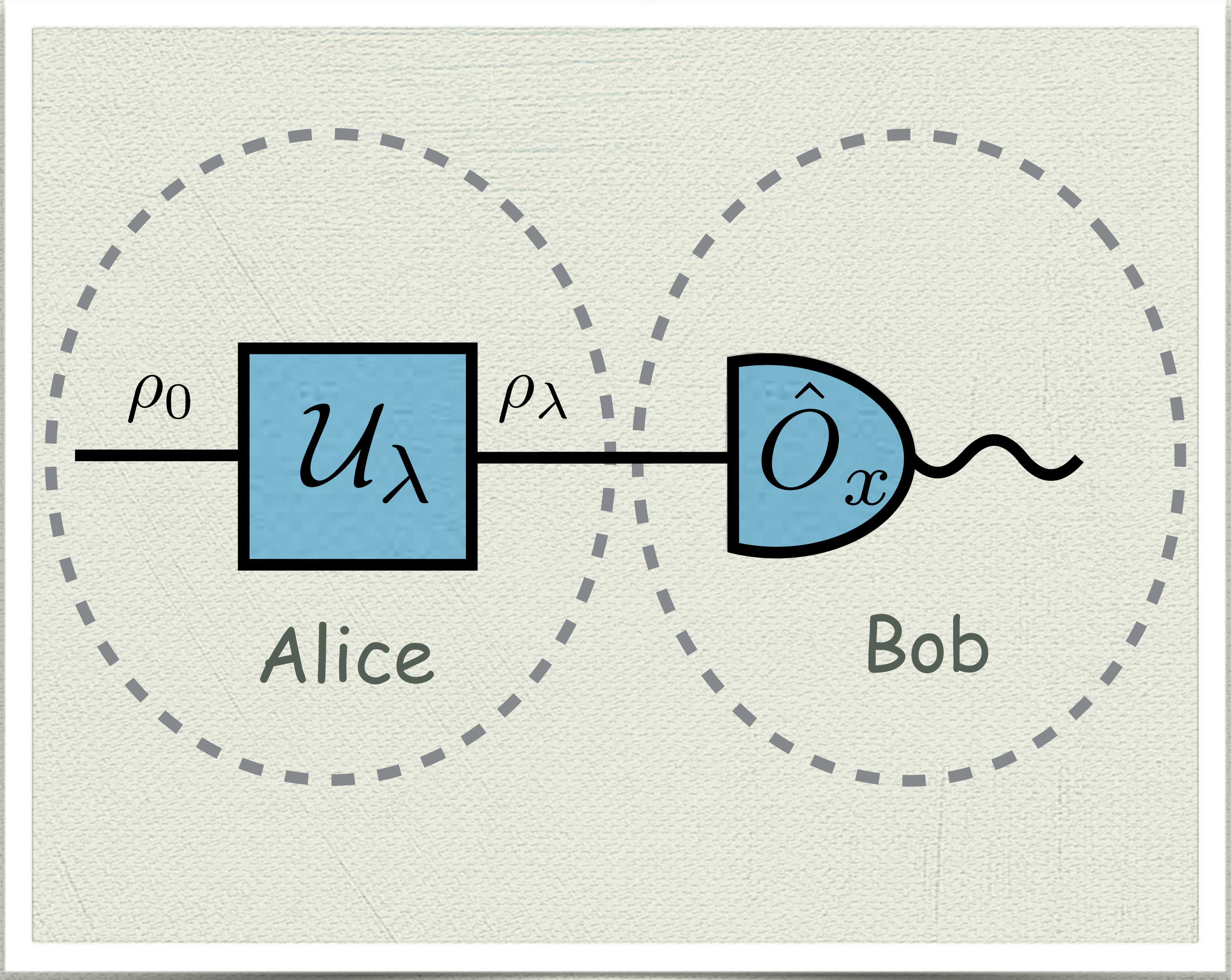
$$\langle(\Delta\hat{\lambda})^2\rangle \geq \frac{1}{NF(\lambda)}$$

$$F(\rho_\lambda) = \int dx \frac{1}{p(x|\lambda)}\left[\frac{d\,p(x|\lambda)}{d\lambda}\right]^2$$

$$p(x|\lambda) = \mathrm{Tr}[\hat{O}_x\rho_\lambda]$$

Quantum CRB

$$N\langle(\Delta\hat{\lambda})^2\rangle \geq \frac{1}{F(\rho_\lambda)} \geq \frac{1}{H(\rho_\lambda)}$$

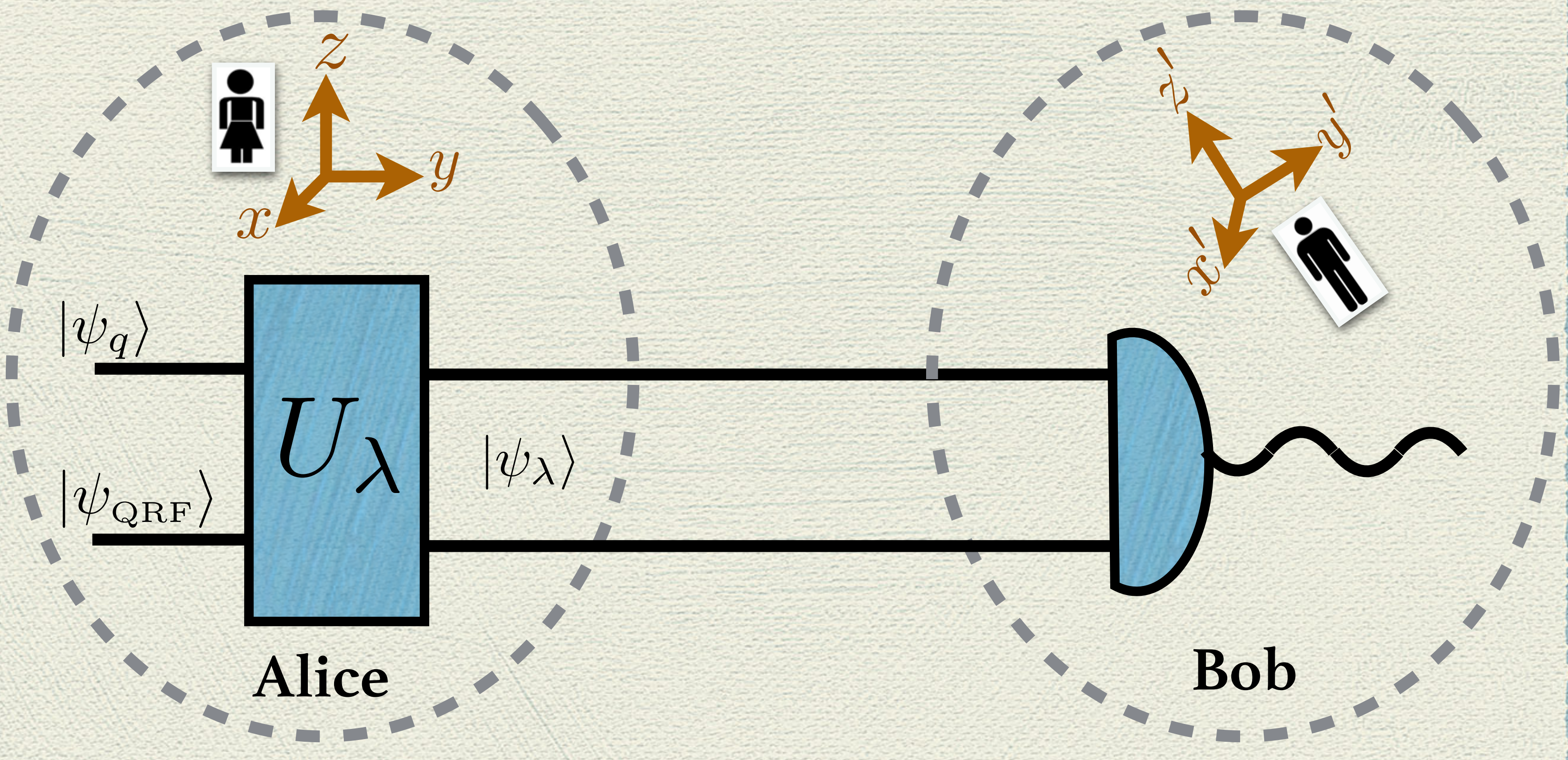


$$|\psi_0\rangle = |\psi_q\rangle \otimes |\psi_{QRF}\rangle$$

$$|\psi_\lambda\rangle = U_\lambda |\psi_0\rangle$$

$$U_\lambda = e^{i\hat{K}\lambda}$$

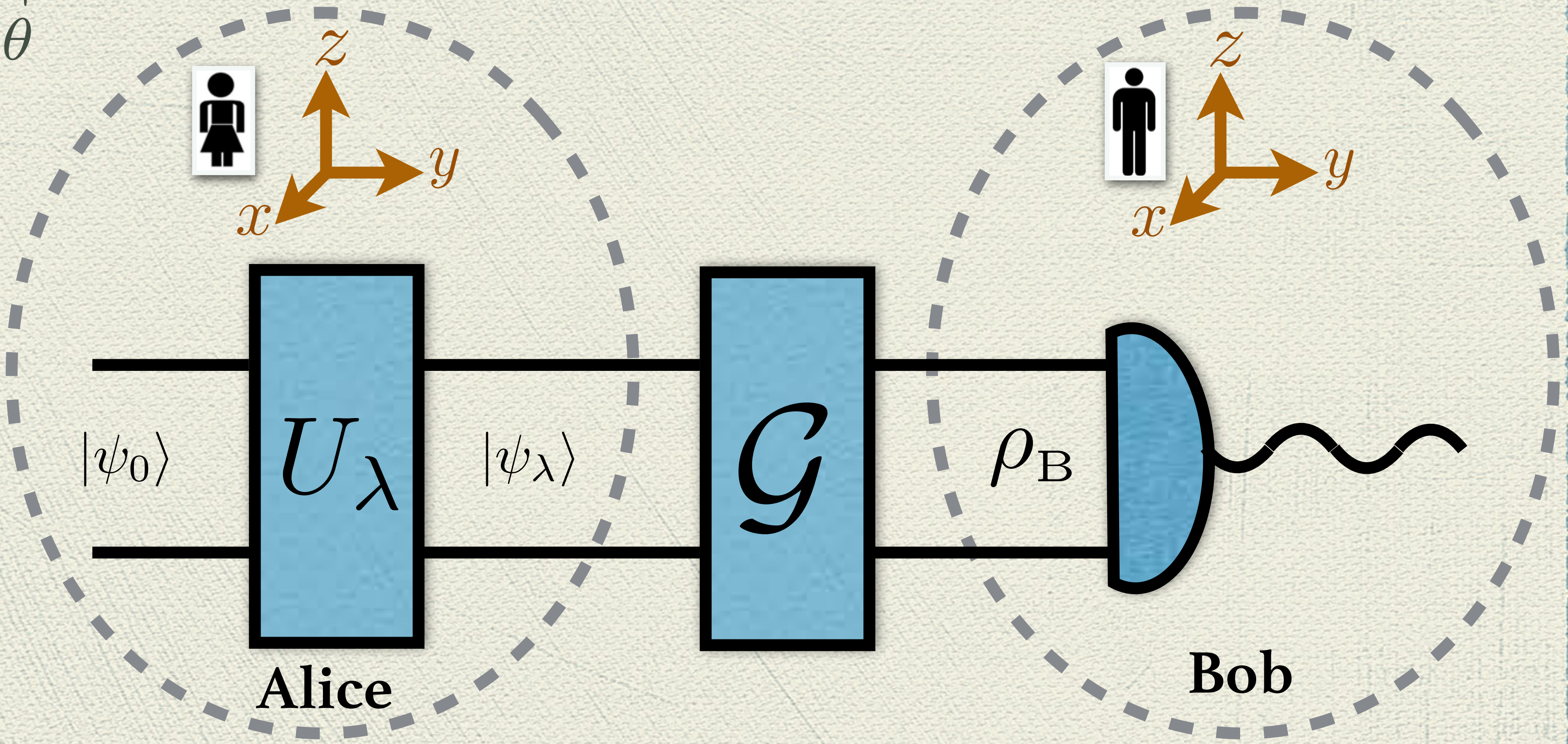
The generator of Alice's encoding



$$\rho_B = \int d\theta \, U_\theta \rho_A U_\theta^\dagger$$

$$U_\theta = e^{i\hat{G}\theta}$$

The generator of the group that relates Alice's local RF to Bob's local RF



Efficiency of communication

Quantum Fisher information loss:

$$\boxed{l(\rho_\lambda, \hat{G}) = H(\rho_\lambda) - H(\mathcal{G}[\rho_\lambda])}$$

$$\boxed{l(\rho, \hat{G}) = 4\sum_i \frac{(\mathrm{Cov}_\rho(\hat{P}_i, \hat{K}))^2}{p_i} \quad \text{where} \quad \hat{G} = \sum_i g_i \hat{P}_i}$$

$$\mathrm{Cov}_\rho(\hat{A}, \hat{B}) = \frac{1}{2}\langle\{\hat{A} - \langle\hat{A}\rangle, \hat{B} - \langle\hat{B}\rangle\}\rangle_\rho = \frac{1}{2}\langle\{\hat{A}, \hat{B}\}\rangle_\rho - \langle\hat{A}\rangle_\rho\langle\hat{B}\rangle_\rho$$

Necessary and sufficient condition for no loss in precision:

$$\boxed{l(\rho, \hat{G}) = 0 \;\Leftrightarrow\; \forall i,\; \mathrm{Cov}_\rho(\hat{P}_i, \hat{K}) = 0}$$

$$\boxed{l(\rho, \hat{G}) = 0 \;\Rightarrow\; \mathrm{Cov}_\rho(\hat{G}, \hat{K}) = 0}$$

Necessary condition for maximum loss:

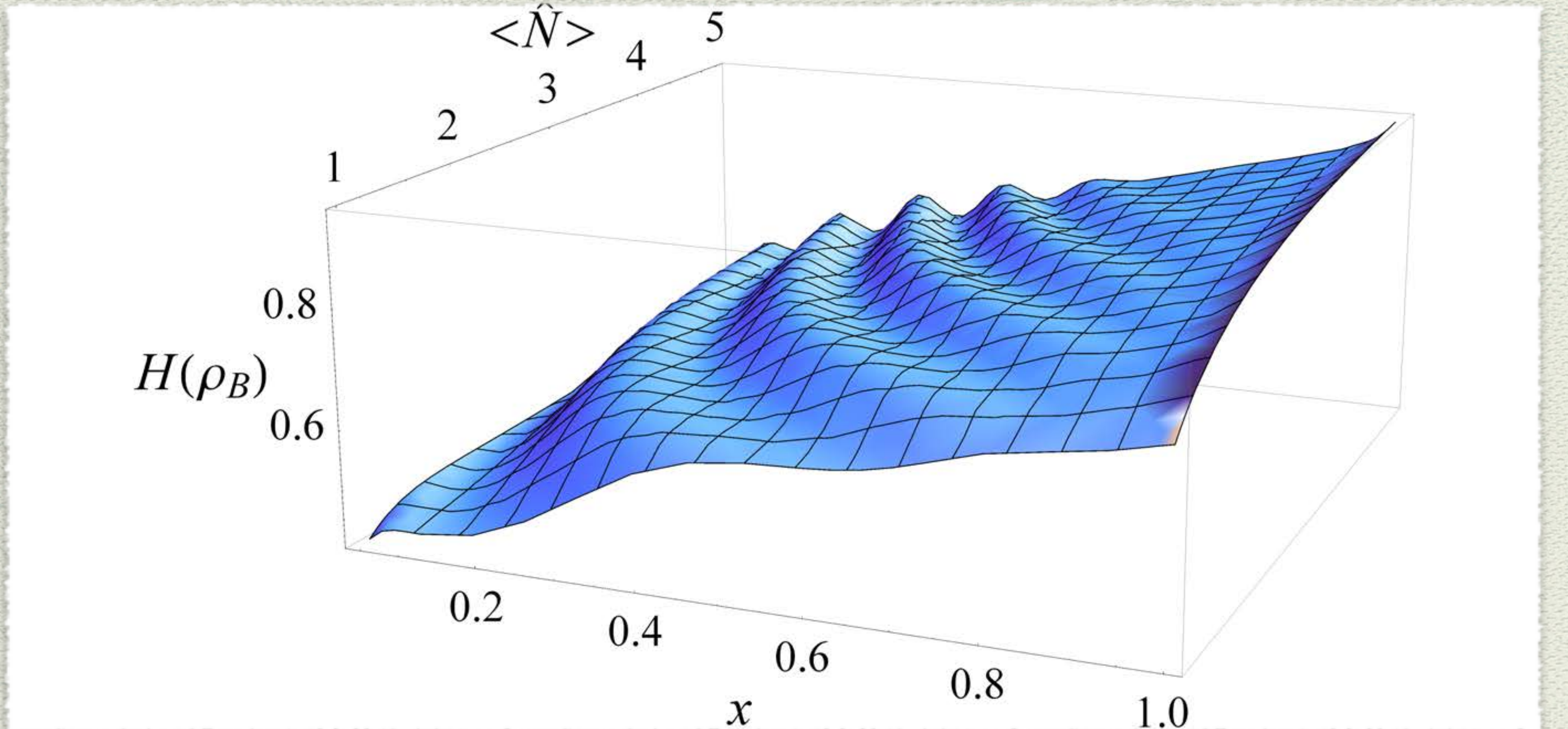
$$\boxed{l(\rho, \hat{G}) = H(\rho) \;\Rightarrow\; \langle [\hat{G}, \hat{K}] \rangle_\rho = 0}$$

Two different cases:

$$[\hat{G}, \hat{K}] = 0$$

$$[\hat{G}, \hat{K}] \neq 0$$

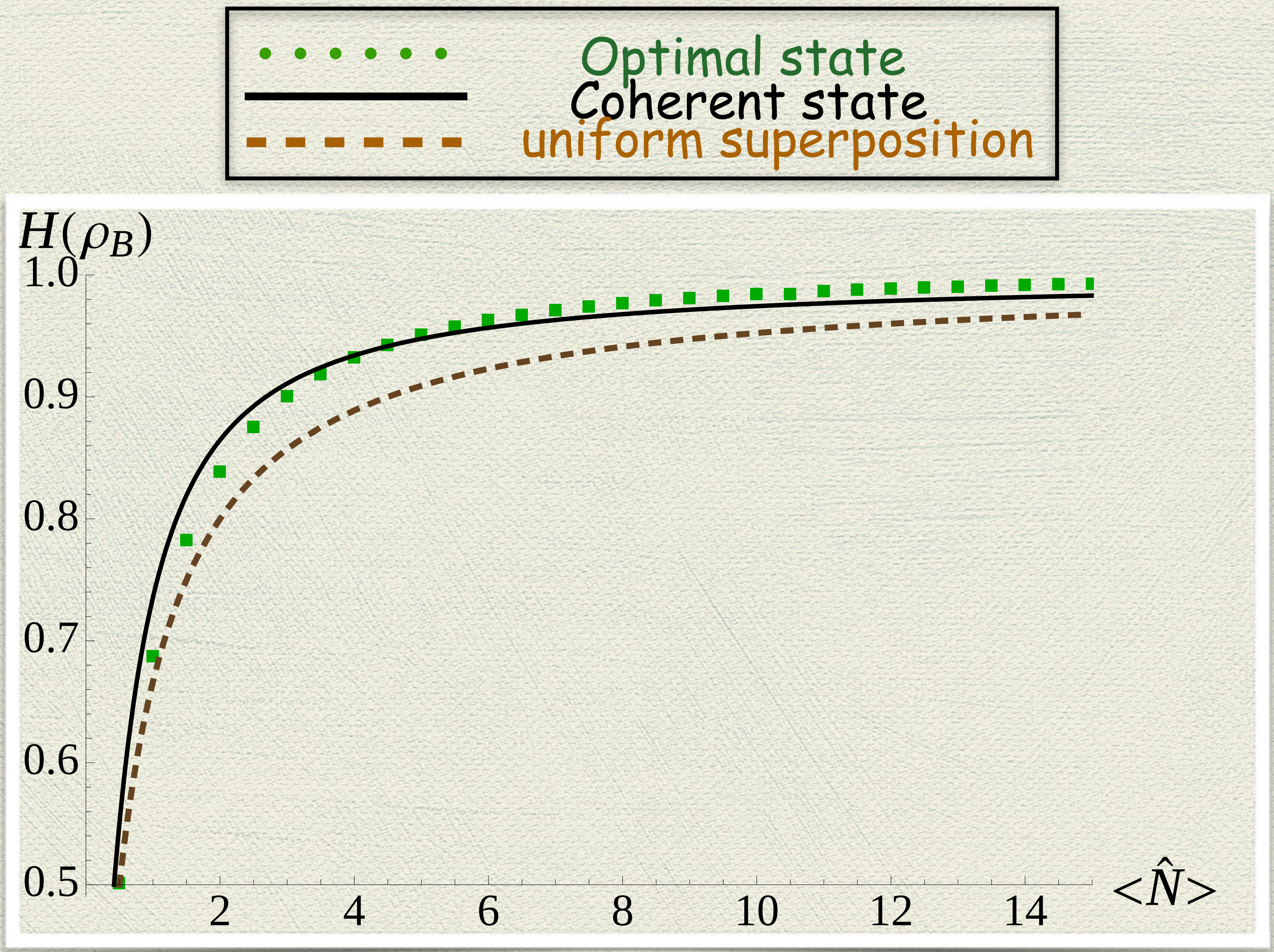
Two non-interacting quantum harmonic oscillators (Commuting case)



$H = \hbar\omega(a^\dagger a + b^\dagger b)$ $\quad \hat{K} = \hat{N}_q = a^\dagger a$ $\quad [\hat{K}, \hat{H}] = 0$ $\quad x = \frac{\alpha^2}{\langle \hat{N} \rangle}$

$|\psi_0\rangle = |\psi_q\rangle \otimes |\psi_{QRF}\rangle$ $\quad |\psi_q\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ $\quad \langle \hat{N} \rangle = \alpha^2 + \sinh^2 r$

The optimal state in this class of states is the coherent state. In fact it can be easily verified analytically that squeezed state give QFI=0.



$$|\psi_{QRF}\rangle = \sum_{n=0}^{N-1} c_n |n\rangle \qquad H(\rho_B) = 2 - 2\left(\sum_{n=0}^{N-2} \frac{|c_n|^4}{|c_n|^2 + |c_{n+1}|^2} + |c_{N-1}|^2\right)$$

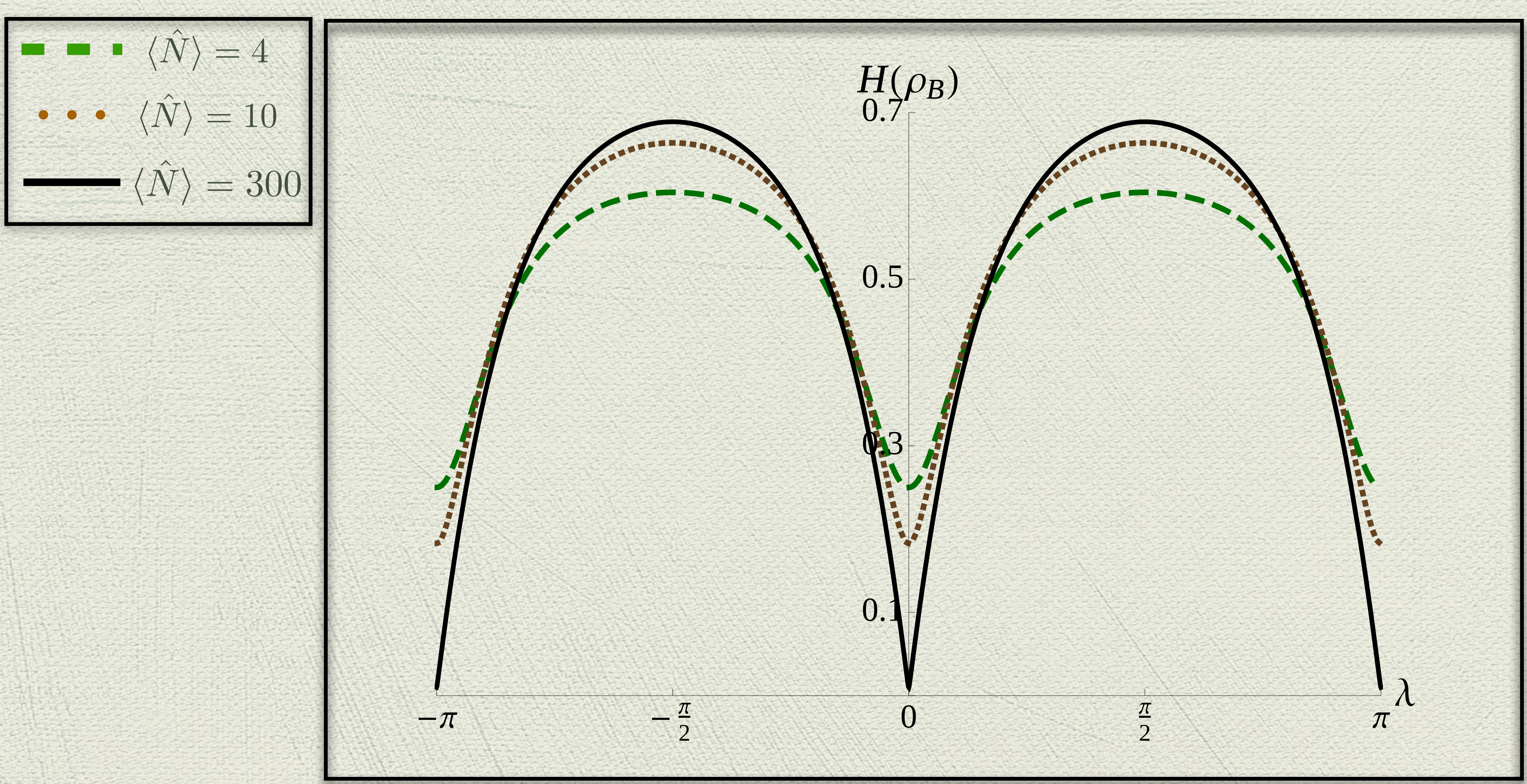
$$c_n = \frac{1}{\sqrt{N}} \rightarrow H_{US}(\rho_B) = 1 - \frac{1}{N}$$

M. A., D. Jennings, and T. Rudolph, New J. Phys. 15, 013057 (2013).

Two interacting quantum harmonic oscillators (non-Commuting case)

$$\hat{H} = \hbar\omega(a^\dagger a + b^\dagger b) + \hbar\kappa(a^\dagger b + b^\dagger a) \qquad [\hat{K}, \hat{H}] \neq 0$$

No decoherence-free subspace: $\forall P, R \in \mathbb{Z},\ P\omega \neq R\kappa$



Covariant vs. non-covariant noise

Covariant case:

- QFI is independent of the parameter to be estimated
- Decoherence-free subsystem is necessary

Non-covariant case:

- QFI depends on the parameter to be estimated
- Estimation is possible even in the absence of DFS

Thanks for your attention!

$$|\Psi\rangle = \frac{1}{\sqrt{M+1}}(|0\rangle + |1\rangle + \cdots + |M\rangle)$$

$$\mathcal{G}[|\Psi_\pm\rangle\langle\Psi_\pm|] = \frac{1}{2(M+1)}(|00\rangle\langle 00| + |1\pm\rangle\langle 1\pm| + \cdots + |M\pm\rangle\langle M\pm| + |M1\rangle\langle M1|)$$

$$|n\pm\rangle = |0\,n\rangle \pm |1\,n-1\rangle$$

$$\boxed{p_? = \mathrm{Tr}[P_?\mathcal{G}(|\Psi_\pm\rangle\langle\Psi_\pm|)] = \frac{1}{M+1}}$$

Increasing the number of terms in the expansion of the state of the QRF leads to increasing the number of decoherence-free subspaces which then increases the probability of successfully discriminating between the two states.

Different quality measures for the performance of a QRF

- Relative entropy of frameness
- Characteristic function: This is more relevant to the efficiency of alignment-free quantum communication
- Our measure is “Quantum Fisher Information loss"

$$\forall g \in G \quad \mathcal{E}[U(g)\rho U^\dagger(g)] = U(g)\mathcal{E}[\rho]U^\dagger(g)$$

$$\forall g \in G \quad [\rho, U(g)] = 0$$

$$H_U(\rho) = 4\big(\langle \hat{K}^2\rangle_\rho - \langle \hat{K}\rangle_\rho^2\big) = 4\mathrm{Var}_\rho(\hat{K})$$